Multiplicative stochastic processes involving the time derivative of a Markov process

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The characteristic functional of the derivative $\dot{\phi}(t)$ of a Markov process $\phi(t)$ and the related multiplicative process $\sigma(t)$, which obeys the stochastic differential equation $i\dot{\sigma}(t) = (A + \dot{\phi}(t)B)\sigma(t)$, have been studied. Exact equations for the marginal characteristic functional and the marginal average of $\sigma(t)$ are derived. The first equation is applied to obtain a set of equations for the marginal moments of $\dot{\phi}(t)$ in terms of the prescribed properties of $\phi(t)$. It is illustrated by an example how these equations can be solved, and it is shown in general that $\dot{\phi}(t)$ is delta correlated, with a smooth background. The equation of motion for the marginal average of $\sigma(t)$ is solved for various cases, and it is shown how closed-form analytical expressions for the average $\langle\sigma(t)\rangle$ can be obtained.

I. INTRODUCTION

The equation of motion for the density operator of an atom in a finite-bandwidth laser field or the equation for the regression of the atomic dipole correlations assumes the general form[1,2]

$$i\frac{d\sigma}{dt} = (A + \dot{\phi}(t)B)\sigma, \tag{1.1}$$

where A and B are linear operators in Liouville space, which act on the Liouville vector $\sigma(t)$. Here $\phi(t)$ represents the laser phase, which is considered to be a real-valued stochastic process. The fluctuating phase broadens the laser line, but the atom responds to the instantaneous frequency shift $\dot{\phi}(t)$, which is the time derivative of the laser phase.[3] The process $\dot{\phi}(t)$ is again a stochastic process, and via Eq. (1.1) the state of the atom or the correlation functions $\sigma(t)$ become stochastic quantities. The issue in quantum optics is then to solve the multiplicative stochastic differential equation (1.1) for the average $\langle\sigma(t)\rangle$. The first solution was obtained by Fox,[4] who assumed the process $\dot{\phi}(t)$ to be Gaussian white noise, which corresponds to a diffusive Gaussian phase $\phi(t)$ (the Wiener–Lévy process). This result was generalized to a Gaussian process $\dot{\phi}(t)$ with a finite correlation time and an exponentially decaying correlation function[5–7] (the Ornstein–Uhlenbeck process), and to a process $\phi(t)$, which is again diffusive, but not Gaussian[8,9] (the independent-increment process). Furthermore, Eq. (1.1) can be solved for $\langle\sigma(t)\rangle$ if we have $\dot{\phi}(t)$ as a Markov random-jump process,[10–13] which models a multimode laser.[14,15]

In these examples the solvability of the problem relies on the Gaussian property of $\dot{\phi}(t)$, or hinges on the prescribed stochastics of $\dot{\phi}(t)$. This implies that the process $\dot{\phi}(t)$ is actually considered to be the driving process. For a single-mode laser in general, however, the phase fluctuations $\phi(t)$ are specified rather than the derivative $\dot{\phi}(t)$ of this process. A prime example would be the atomic response to phase-locked radiation,[16] as it is generated for instance by some ring lasers.[17] In this paper we shall develop a general method to solve Eq. (1.1) for the case that $\phi(t)$ is a given Markov process. The formal theory will be exemplified by a specific choice for $\phi(t)$, which models phase-locked radiation. Furthermore, we shall study the time derivative of $\phi(t)$ itself and extract the stochastics of $\dot{\phi}(t)$ from the properties of $\phi(t)$.

II. THE STOCHASTICS OF $\dot{\phi}(t)$

Let us define the phase $\phi(t)$ as a homogeneous Markov process.[18] Then its stochastics is fixed by the probability distribution $P(\phi,t)$ and the conditional probability distribution $P_\tau(\phi_2|\phi_1)$ $(\tau\geqslant 0)$, which has the significance of the probability density for the occurrence of $\phi(t+\tau) = \phi_2$ if $\phi(t) = \phi_1$. For a homogeneous process this is independent of t by definition. The higher-order statistics is now determined by the Markov property.[19] From the obvious relation

$$\int d\phi'\, P_{t-t_0}(\phi|\phi')P(\phi',t_0) = P(\phi,t), \quad t\geqslant t_0, \tag{2.1}$$

it follows that it is sufficient to prescribe the probability distribution $P(\phi,t)$ for a single time point t_0 only. The time evolution towards $t > t_0$ can then be found from Eq. (2.1) and $P_{t-t_0}(\phi|\phi')$.

The conditional probability distribution obeys the Master equation[18]

$$\frac{\partial}{\partial\tau}P_\tau(\phi_3|\phi_1) = \int d\phi_2\{W(\phi_3|\phi_2) - a(\phi_2)\delta(\phi_3-\phi_2)\}P_\tau(\phi_2|\phi_1), \tag{2.2}$$

with $W(\phi'|\phi)\geqslant 0$ as the transition rate of the process from ϕ to ϕ' and

$$a(\phi) = \int d\phi'\, W(\phi'|\phi), \tag{2.3}$$

which is the loss rate of ϕ, independent of the final value ϕ'. The initial condition for Eq. (2.2) reads

$$P_0(\phi_3|\phi_1) = \delta(\phi_3 - \phi_1), \tag{2.4}$$

so a given $W(\phi'|\phi)$ determines $P_\tau(\phi_3|\phi_1)$ for every $\tau\geqslant 0$.

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Hence the stochastics of a homogeneous Markov process $\phi(t)$ is fixed as soon as $P(\phi,t_0)$ and $W(\phi'|\phi)$ are prescribed. These functions will from now on be assumed to be given.

III. THE CHARACTERISTIC FUNCTIONAL

A convenient way to represent the stochastic properties of a stochastic process is by means of its characteristic functional.[9,20] Since we are concerned with the process $\dot{\phi}(t)$, we define

$$Z_t[k] = \left\langle \exp\left(-i\int_{t_0}^{t} ds\,\dot{\phi}(s)k(s)\right)\right\rangle, \quad t \geqslant t_0, \tag{3.1}$$

which is a functional of the test function $k(t)$. Here the angle brackets denote an average over the stochastic process $\phi(t)$ or $\dot{\phi}(t)$, whatever is prescribed. A general method to evaluate $Z_t[k]$ for the case where $\dot{\phi}(t)$ is a homogeneous Markov process has been given by van Kampen.[21]

Knowledge of the characteristic functional $Z_t[k]$ determines completely the stochastics of $\dot{\phi}(t)$, which can be seen as follows. Choose $k(s)$ as the sequence of δ functions

$$k(s) = -\sum_{l=1}^{n} \delta(s-t_l)k_l, \quad t_l > t_0, \tag{3.2}$$

and take $t=\infty$ in (3.1). Then we find

$$Z_\infty[k] = \langle \exp(ik_n\dot{\phi}(t_n) + \cdots + ik_1\dot{\phi}(t_1))\rangle, \tag{3.3}$$

which is the moment-generating function of $\dot{\phi}(t)$. If we write $z_n(k_n,t_n;...;k_1,t_1)$, then we can obtain the moments of $\psi(t)\equiv\dot{\phi}(t)$ according to

$$\langle\psi(t_n)\cdots\psi(t_1)\rangle = (-i)^n \frac{\partial}{\partial k_n}\cdots\frac{\partial}{\partial k_1} \times z_n(k_n,t_n;...;k_1,t_1)\Big|_{k_n=\cdots=k_1=0}, \tag{3.4}$$

and the probability distributions by

$$\bar{P}_n(\psi_n,t_n;...;\psi_1,t_1) = \frac{1}{(2\pi)^n}\int dk_n\cdots dk_1 \times e^{-ik_n\psi_n-\cdots-ik_1\psi_1} z_n(k_n,t_n;...;k_1,t_1), \tag{3.5}$$

where we have introduced $\bar{P}_n$ in order to distinguish from the probability distributions for $\phi(t)$ itself.

IV. THE MARGINAL AVERAGE

A. General

The exponential in Eq. (3.1) is a functional of both $k(t)$ and $\dot{\phi}(t)$, so it depends on the values of $\dot{\phi}(t)$ in the complete interval $[t_0,t]$. After the average has been taken it will be only a functional of $k(t)$. The general attempt to evaluate averages of a functional is to derive an equation for the average. For subsequently solving this equation for functionals which involve Markov processes, this scheme is most conveniently carried out by an intermediate introduction of Burshtein's marginal averages.[22] Since in our problem the stochastics of $\phi(t)$ is assumed to be given, the appropriate marginal characteristic functional, which is related to $Z_t[k]$, should be defined as

$$Q_t[\phi_0,k] = \left\langle \delta(\phi(t)-\phi_0)\exp\left(-i\int_{t_0}^{t} ds\,\dot{\phi}(s)k(s)\right)\right\rangle, \tag{4.1}$$

for $t\geqslant t_0$. The initial value is then

$$Q_{t_0}[\phi_0,k] = \langle\delta(\phi(t_0)-\phi_0)\rangle = P(\phi_0,t_0), \tag{4.2}$$

and $Z_t[k]$ follows from $Q_t[\phi_0,k]$ according to

$$Z_t[k] = \int d\phi_0\, Q_t[\phi_0,k]. \tag{4.3}$$

For $t=t_0$ we find with Eq. (4.2)

$$Z_{t_0}[k] = \int d\phi_0\, P(\phi_0,t_0) = 1,$$

in agreement with Eq. (3.1).

In order to derive an equation for the time evolution of $Q_t[\phi_0,k]$, we first increase t by a small amount $\Delta t>0$. This gives

$$Q_{t+\Delta t}[\phi_0,k] = \Big\langle \delta(\phi(t+\Delta t)-\phi_0) \times \exp\{-i(\phi(t+\Delta t)-\phi(t))k(t)\} \times \exp\left(-i\int_{t_0}^{t} ds\,\dot{\phi}(s)k(s)\right)\Big\rangle. \tag{4.4}$$

Subsequently, we expand the exponential functional of $\dot{\phi}(s)$ in a series, and we take the average in (4.4) term by term. Hereafter, we apply the Master equation (2.2) for $P_{t+\Delta t}(\phi|\phi_0)$ and take the limit $\Delta t\to 0$. This yields an equation for the marginal average, and explicitly we find

$$\frac{\partial}{\partial t}Q_t[\phi_0,k] = \int d\phi\{W(\phi_0|\phi) - a(\phi)\delta(\phi_0-\phi)\} \times e^{-i(\phi_0-\phi)k(t)}\,Q_t[\phi,k]. \tag{4.5}$$

The Markov process $\phi(t)$ is characterized by $P(\phi_0,t_0)$ and $W(\phi_0|\phi)$, which, respectively, determine the initial value and the time evolution of $Q_t[\phi_0,k]$. For a specific choice of $W(\phi_0|\phi)$, we have to solve Eq. (4.5), after which the characteristic functional $Z_t[k]$ can be obtained from Eq. (4.3).

Notice the resemblance between the result (4.5) and the Master equation (2.2). If we multiply Eq. (2.2) by $P(\phi_1,t_0)$, take $\tau=t-t_0$ and apply the relation (2.1), we find

$$\frac{\partial}{\partial t}P(\phi_0,t) = \int d\phi\{W(\phi_0|\phi)-a(\phi)\delta(\phi_0-\phi)\}P(\phi,t), \tag{4.6}$$

which is the Master equation for $P(\phi_0,t)$. This equation is identical to Eq. (4.5), including the initial condition (4.2), if we set $k(t)\equiv 0$. On the other hand, it follows from Eq. (4.1) that $Q_t[\phi_0,k] = \langle\delta(\phi(t)-\phi_0)\rangle = P(\phi_0,t)$ if we take $k(t)=0$, so that in this case Eq. (4.5) should indeed reduce to Eq. (4.6).

B. Independent increments

In order to display the usefulness and applicability of the marginal-functional approach, we consider an example. Let us specify the transition rate by

$$W(\phi_0|\phi) = \gamma w(\phi_0-\phi), \quad \gamma>0, \tag{4.7}$$

where the function $w(\eta)$ is normalized as

$$\int d\eta \, w(\eta) = 1 . \tag{4.8}$$

The stochastic process $\phi(t)$ will be defined on the real axis, with $-\infty < \phi < \infty$. The assertion (4.7) states that the probability for a transition $\phi \to \phi_0$ depends only on the phase difference $\phi_0 - \phi$, and from Eq. (2.3) we find that $a(\phi) = \gamma$, so that the total loss rate for ϕ is independent of ϕ. This is a diffusion process, and it is commonly referred to as the independent-increment process. As an initial condition for the probability distribution, we take

$$P(\phi,t_0) = \delta(\phi) . \tag{4.9}$$

Comparison of the Master equations for $P_\tau(\phi|\phi')$ and $P(\phi,t)$ then shows that the probability distribution and the conditional probability distribution are related according to

$$P_{t-t_0}(\phi|\phi_0) = P(\phi - \phi_0,t) . \tag{4.10}$$

The Master equation (4.6) for $P(\phi,t)$ can be solved by Fourier transformation with respect to ϕ. If we write

$$\hat{P}(\rho,t) = \langle e^{i\rho\phi(t)}\rangle = \int_{-\infty}^{\infty} d\phi \, e^{i\rho\phi} P(\phi,t) , \tag{4.11}$$

which has $\hat{P}(\rho,t_0) = 1$ as the initial condition, then the solution of Eq. (4.6) is immediately seen to be

$$\hat{P}(\rho,t) = e^{\gamma(\hat{w}(\rho) - 1)(t-t_0)} , \quad t \geqslant t_0 , \tag{4.12}$$

in terms of the Fourier transform $\hat{w}(\rho)$ of $w(\phi)$. Note that $\hat{w}(0) = 1$, as a result of the normalization (4.7). Along the very same lines we can solve Eq. (4.55) for the Fourier transform $\hat{Q}_t[\rho,k]$. We obtain

$$\hat{Q}_t[\rho,k] = \exp\left(-\gamma \int_{t_0}^{t} ds \int_{-\infty}^{\infty} d\phi (1 - e^{i\phi(\rho - k(s))}) w(\phi) \right), \tag{4.13}$$

after which the characteristic functional follows from

$$Z_t[k] = \hat{Q}_t[0,k] , \tag{4.14}$$

which yields the familiar result.[8]

V. THE MARGINAL MOMENTS

A. General

If we take $k(s)$ as the sequence of delta functions (3.2) in the definition (4.1) of the marginal characteristic functional, it assumes the form

$$Q_t[\phi_0,k] = \left\langle \delta(\phi(t) - \phi_0) \exp\left(i \sum_{l=1}^{n} k_l \psi(t_l) \Theta(t - t_l) \right) \right\rangle , \tag{5.1}$$

with $\psi(t) = \dot{\phi}(t)$ and $\Theta(t)$ the unit-step function. Just as we can find the moments $\langle \psi(t_n) \cdots \psi(t_1) \rangle$ of $\psi(t)$ from $Z_\infty[k]$, we can obtain the marginal moments $\langle \delta(\phi(t) - \phi_0) \psi(t_n) \cdots \psi(t_1) \rangle$ from $Q_t[\phi_0,k]$. Obviously the integral over ϕ_0 of the marginal moments yields the moments. The characteristic functional $Z_t[k]$ becomes independent of t if $t > t_l$ for all l, but $Q_t[\phi_0,k]$ remains time dependent. This is due to the appearance of $\delta(\phi(t) - \phi_0)$. Furthermore, the time t is a dynamical variable in Eq. (4.55), so that care should be exercised in the time ordering. The marginal moments follow from $Q_t[\phi_0,k]$ by differentiation, according to

$$\begin{aligned} &\langle \delta(\phi(t) - \phi_0) \psi(t_n) \cdots \psi(t_1) \rangle \Theta(t - t_n) \cdots \Theta(t - t_1) \\ &= (-i)^n \frac{\partial}{\partial k_n} \cdots \frac{\partial}{\partial k_1} Q_t[\phi_0,k] \bigg|_{k_n = \cdots = k_1 = 0} . \end{aligned} \tag{5.2}$$

Equation (4.5) for $Q_t[\phi_0,k]$ implies an equation for the marginal moments. First, we note that

$$\begin{aligned} &\exp\{ -i(\phi_0 - \phi) k(t) \} Q_t[\phi,k] \\ &= \left\langle \delta(\phi(t) - \phi) \exp\left\langle i \sum_{l=1}^{n} k_l \{ (\phi_0 - \phi) \delta(t - t_l) \right.\right. \\ &\quad \left.\left. + \psi(t_l) \Theta(t - t_l) \} \right\rangle \right\rangle . \end{aligned} \tag{5.3}$$

After substituting this expression in the right-hand side of Eq. (4.5), differentiating with respect to $k_n,...,k_1$, setting $k_n = \cdots = k_1 = 0$, and integrating over time, we obtain

$$\begin{aligned} &\langle \delta(\phi(t) - \phi_0) \psi(t_n) \cdots \psi(t_1) \rangle \Theta(t - t_n) \cdots \Theta(t - t_1) \\ &= \int d\phi \{ W(\phi_0|\phi) - a(\phi) \delta(\phi_0 - \phi) \} \\ &\quad \times \int_{t_0}^{t} dt' \langle \delta(\phi(t') - \phi) \{ (\phi_0 - \phi) \delta(t' - t_n) \\ &\quad + \psi(t_n) \Theta(t' - t_n) \} \\ &\quad \cdots \{ (\phi_0 - \phi) \delta(t' - t_1) + \psi(t_1) \Theta(t' - t_1) \} \rangle . \end{aligned} \tag{5.4}$$

When we set $t > t_l$ for all l, we have a Master-like equation for $\langle \delta(\phi(t) - \phi_0) \psi(t_n) \cdots \psi(t_1) \rangle$, and the lower-order marginal moments $\langle \delta(\phi(t) - \phi_0) \psi(t_m) \cdots \psi(t_1) \rangle$ with $m < n$ appear as inhomogeneous terms. Hence Eq. (5.4) should be solved successively for $n = 1$, $n = 2$,... . We note that Eq. (5.4) provides an explicit expression for $\langle \psi(t_n) \cdots \psi(t_1) \rangle$ in terms of the lower-order marginal moments after an integration over ϕ_0. Indeed, from the property

$$\int d\phi_0 \{ W(\phi_0|\phi) - a(\phi) \delta(\phi_0 - \phi) \} = 0 , \tag{5.5}$$

the term with $\langle \delta(\phi(t') - \phi) \psi(t_n) \cdots \psi(t_1) \rangle$ on the right-hand side of Eq. (5.4) vanishes after an integration over ϕ_0.

B. Lowest orders

In order to exhibit clearly the structure of the equation for the marginal moments, we consider the cases $n = 1$ and $n = 2$ in some more detail. After a slight rearrangement, Eq. (5.4) for $n = 1$ can be written as

$$\begin{aligned} &\langle \delta(\phi(t) - \phi_0) \psi(t_1) \rangle \\ &= \int d\phi \{ W(\phi_0|\phi) - a(\phi) \delta(\phi_0 - \phi) \} \\ &\quad \times \left\{ (\phi_0 - \phi) P(\phi,t_1) \right. \\ &\quad \left. + \int_{t_1}^{t} dt' \langle \delta(\phi(t') - \phi) \psi(t_1) \rangle \right\} , \end{aligned} \tag{5.6}$$

for $t \geqslant t_1$. This integral equation in time is equivalent to the differential equation

$$\frac{\partial}{\partial t}\langle\delta(\phi(t)-\phi_0)\psi(t_1)\rangle$$
$$=\int d\phi\{W(\phi_0|\phi)-a(\phi)\delta(\phi_0-\phi)\}$$
$$\times\langle\delta(\phi(t)-\phi)\psi(t_1)\rangle\,, \tag{5.7}$$

together with the initial condition

$$\langle\delta(\phi(t_1)-\phi_0)\psi(t_1)\rangle$$
$$=\int d\phi\{W(\phi_0|\phi)-a(\phi)\delta(\phi_0-\phi)\}$$
$$\times(\phi_0-\phi)P(\phi,t_1)\,. \tag{5.8}$$

The equation for the first marginal average $\langle\delta(\phi(t)-\phi_0)\psi(t_1)\rangle$ is identical to the Master equation (2.2), but with a different initial value.

Integration of (5.8) over ϕ_0 yields

$$\langle\psi(t_1)\rangle=\int d\phi\int d\phi_0\, W(\phi_0|\phi)(\phi_0-\phi)P(\phi,t_1)\,, \tag{5.9}$$

which expresses explicitly the average of $\langle\psi(t_1)\rangle$ in the given functions $W(\phi_0|\phi)$ and $P(\phi,t_1)$. With the aid of the Master equation, we can cast (5.9) in the form

$$\langle\psi(t_1)\rangle=\int d\phi\,\phi\frac{\partial}{\partial t_1}P(\phi,t_1)=\frac{d}{dt_1}\langle\phi(t_1)\rangle\,, \tag{5.10}$$

as it should be.

The solution of Eq. (5.6) for $\langle\delta(\phi(t)-\phi_0)\psi(t_1)\rangle$ provides the input for the explicit expression for the two-time correlation function, which becomes

$$\langle\psi(t_2)\psi(t_1)\rangle$$
$$=\int d\phi\int d\phi_0\{W(\phi_0|\phi)-a(\phi)\delta(\phi_0-\phi)\}$$
$$\times\{(\phi_0-\phi)^2\delta(t_2-t_1)P(\phi,t_2)$$
$$+(\phi_0-\phi)(\langle\delta(\phi(t_2)-\phi)\psi(t_1)\rangle\Theta(t_2-t_1)$$
$$+\langle\delta(\phi(t_1)-\phi)\psi(t_2)\rangle\Theta(t_1-t_2))\}\,. \tag{5.11}$$

The appearance of $\delta(t_2-t_1)$ shows that the time derivative of any Markov process is δ correlated with a continuous background.

C. Random jumps

Equation (5.11) for instance might seem awkward, but it is really straightforward in its application. Let us illustrate this with an example. Consider the random-jump process $\phi(t)$, defined as a stationary process with transition rate

$$W(\phi|\phi')=\gamma P(\phi)\,,\quad \gamma>0\,, \tag{5.12}$$

in terms of an arbitrary probability distribution $P(\phi)$. Equation (5.12) is equivalent to the statement that the probability for a transition $\phi'\to\phi$ is independent of the initial value ϕ' (see Ref. 13). From Eq. (5.9) we immediately derive

$$\langle\psi(t_1)\rangle=0\,, \tag{5.13}$$

which is, in view of (5.10), necessary for a stationary process. From (2.3) we obtain $a(\phi)=\gamma$, and the solution of Eq. (5.7), with initial value (5.8), is readily found to be

$$\langle\delta(\phi(t)-\phi_0)\psi(t_1)\rangle=\gamma P(\phi_0)(\phi_0-b_1)e^{-\gamma(t-t_1)}\,,\quad t\geqslant t_1\,. \tag{5.14}$$

Here we have introduced the moments of $P(\phi)$ as

$$b_n=\int d\phi\,\phi^nP(\phi)\,, \tag{5.15}$$

which are parameters of the process $\phi(t)$. Solution (5.14) can be substituted into Eq. (5.11), which gives the correlation function

$$\langle\psi(t_1)\psi(t_2)\rangle=\gamma(b_2-b_1^2)\{2\delta(t_1-t_2)-\gamma e^{-\gamma|t_2-t_1|}\}\,, \tag{5.16}$$

for all t_1,t_2. From (5.15) it follows that

$$b_2-b_1^2\geqslant 0\,, \tag{5.17}$$

so that for $t_1\neq t_2$ the correlation (5.16) is negative. For $t_1=t_2$ the δ function dominates the negative term, so that $\langle\psi(t_1)^2\rangle$ is positive, as it should be.

VI. THE MULTIPLICATIVE PROCESS

So far we have considered the stochastics of $\dot\phi(t)$ itself, and its characteristic functional. In this section we shall generalize the method, in order to solve the multiplicative equation (1.1). To this end we write the formal solution of (1.1) for the stochastic vector $\sigma(t)$ as

$$\sigma(t)=e^{-iA(t-t_0)}T\exp\Big[-i\int_{t_0}^t ds\,\dot\phi(s)\widetilde{B}(s)\Big]\sigma(t_0)\,, \tag{6.1}$$

where T is the time-ordering operator and $\widetilde{B}(t)$ is defined as

$$\widetilde{B}(t)=e^{iA(t-t_0)}Be^{-iA(t-t_0)}\,. \tag{6.2}$$

In close analogy to the definition of $Q_t[\phi_0,k]$ in Eq. (4.1), we now introduce the marginal average of $\sigma(t)$ by

$$\zeta(\phi_0,t)=\langle\delta(\phi(t)-\phi_0)\sigma(t)\rangle\,. \tag{6.3}$$

Then we substitute the expression (6.1) for $\sigma(t)$ and replace t by $t+\Delta t$, which gives a formula similar to Eq. (4.4). That this can also be done for the time-ordered exponential is sometimes referred to as the semigroup property of the evolution operator. Along the same lines that led to Eq. (4.5) we now find

$$i\frac{\partial}{\partial t}\zeta(\phi_0,t)=A\zeta(\phi_0,t)+i\int d\phi\{W(\phi_0|\phi)$$
$$-a(\phi)\delta(\phi_0-\phi)\}e^{-i(\phi_0-\phi)B}\zeta(\phi,t)\,, \tag{6.4}$$

or equivalently

$$\Big(i\frac{\partial}{\partial t}-A+ia(\phi_0)\Big)\zeta(\phi_0,t)$$
$$=i\int d\phi\,W(\phi_0|\phi)e^{-i(\phi_0-\phi)B}\zeta(\phi,t)\,. \tag{6.5}$$

Notice that the operator B appears in the exponential, rather than $\widetilde{B}(t)$, as could be expected by analogy with the characteristic functional. For a given stochastic process $\phi(t)$, e.g., a given $W(\phi|\phi')$ and $P(\phi,t_0)$, we have to solve Eq. (6.4) with the initial condition

$$\zeta(\phi_0,t_0) = \langle \delta(\phi(t_0) - \phi_0)\sigma(t_0) \rangle \,, \tag{6.6}$$

after which $\langle \sigma(t) \rangle$ follows from

$$\langle \sigma(t) \rangle = \int d\phi_0 \, \zeta(\phi_0,t) \,. \tag{6.7}$$

For a given nonstochastic state $\sigma(t_0)$, the initial condition reduces to

$$\zeta(\phi_0,t_0) = P(\phi_0,t_0)\sigma(t_0) \,, \tag{6.8}$$

which differs from (6.6) by the fact that there are no initial correlations. This means that the process $\sigma(t)$ has no memory to times smaller than t_0, and consequently its evolution for $t \geqslant t_0$ is completely determined by its initial state $\sigma(t_0)$. It was emphasized by Arnoldus and Nienhuis[13] that the common choice $\zeta(\phi_0,t_0) = P(\phi_0,t_0)\langle \sigma(t_0) \rangle$ is merely an approximation which only holds for small correlation times of $\dot{\phi}(t)$.

VII. SOLUTIONS

A. Independent increments

Equation (6.5) for the marginal average of $\sigma(t)$ can be solved for the independent-increment process with the same procedure as in Sec. IV, where we obtained the characteristic functional. If we adopt the Fourier transform

$$\hat{\zeta}(\rho,t) = \int_{-\infty}^{\infty} d\phi \, e^{i\rho\phi} \zeta(\phi,t) = \langle e^{i\rho\phi(t)} \sigma(t) \rangle \,, \tag{7.1}$$

where the second equality follows after application of Eq. (6.3), then $\langle \sigma(t) \rangle$ can be found from

$$\langle \sigma(t) \rangle = \hat{\zeta}(0,t) \,. \tag{7.2}$$

With the technique of Sec. IV we can find $\hat{\zeta}(\rho,t)$, and if we differentiate the result with respect to time, we find

$$i \frac{\partial}{\partial t} \hat{\zeta}(\rho,t) = (A - i\hat{W}(\rho))\hat{\zeta}(\rho,t) \,, \tag{7.3}$$

with

$$\hat{W}(\rho) = \gamma \int_{-\infty}^{\infty} d\eta (1 - e^{i\eta(\rho - B)}) w(\eta) \,. \tag{7.4}$$

The operator $\hat{W}(\rho)$ accounts for the phase fluctuations. If we set $\rho = 0$ in Eq. (7.3), we achieve the equation for $\langle \sigma(t) \rangle$, with solution

$$\langle \sigma(t) \rangle = e^{-i(A - i\hat{W}(0))(t - t_0)} \langle \sigma(t_0) \rangle \,, \tag{7.5}$$

for $t \geqslant t_0$. We note that $\langle \sigma(t) \rangle$ can be expressed in terms of $\langle \sigma(t_0) \rangle$ for this process, so that there are no initial correlations for the diffusion process. The process $\dot{\phi}(t)$ has no memory, and with the results of Sec. V it can be shown that $\dot{\phi}(t)$ is indeed delta correlated. This means that $\langle \dot{\phi}(t_n) \cdots \dot{\phi}(t_1) \rangle$ for all n contains only δ functions, which implies the factorization in (7.5).

A special case arises if we take

$$\gamma w(\eta) = \gamma \delta(\eta) + \lambda \delta''(\eta) \,, \quad \lambda > 0 \,, \tag{7.6}$$

where the primes on the δ function denote differentiation with respect to its argument. It is easy to check that this process is the Wiener–Lévy process, or the phase-diffusion process. If we substitute (7.6) into (4.12), we find that $P(\phi,t)$ is Gaussian, and obviously this is the only Gaussian limit of the diffusion process. The operator $\hat{W}(\rho)$ in (7.4) reduces to

$$\hat{W}(\rho) = \lambda(\rho - B)^2 \,, \tag{7.7}$$

and the equation for $\langle \sigma(t) \rangle$ becomes

$$i \frac{d}{dt} \langle \sigma(t) \rangle = (A - i\lambda B^2) \langle \sigma(t) \rangle \,, \tag{7.8}$$

which is the result of Fox.[4]

B. Ornstein–Uhlenbeck process

The diffusion process has no memory and is essentially nonstationary. The initial distribution $P(\phi,t_0) = \delta(\phi)$ diffuses over the whole ϕ axis, $-\infty < \phi < \infty$. The inclusion of a finite memory time can stabilize this process. Let us define the transition probability as

$$\begin{aligned} W(\phi_0|\phi) - a(\phi)\delta(\phi_0 - \phi) &= \lambda \delta''(\phi_0 - \phi) \\ &\quad + \gamma\phi\delta'(\phi_0 - \phi) \,, \quad \lambda > 0 \,, \quad \gamma > 0 \,. \end{aligned} \tag{7.9}$$

Then the Master equation (4.6) for $P(\phi,t)$ becomes the Fokker–Planck equation[18]

$$\frac{\partial}{\partial t} P(\phi,t) = \left(\lambda \frac{\partial^2}{\partial \phi^2} + \gamma \frac{\partial}{\partial \phi} \phi \right) P(\phi,t) \,, \tag{7.10}$$

which has the solution, for $t \to \infty$,

$$P(\phi) = (2\pi\sigma^2)^{-1/2} e^{-\phi^2/2\sigma^2} \,, \quad \sigma^2 = \lambda/\gamma \,. \tag{7.11}$$

This $P(\phi)$, together with $W(\phi_0|\phi)$ from (7.9), defines a stationary Gaussian–Markov process, the Ornstein–Uhlenbeck process. In the limit $\gamma \to 0$ and λ finite (so $\sigma^2 \to \infty$), the process $\phi(t)$ reduces to the Wiener–Lévy process from Sec. VII A. From (7.11) we see that $\phi(t)$ is centered around $\phi = 0$. The distribution is Gaussian with a variance σ^2 around the average $\phi = 0$. The preference for $\phi = 0$ expresses that this process can be considered as a model for phase-locked radiation.

With the specific choice (7.9) for the transition rate, the Master equation (6.5) assumes the form of a second-order partial differential equation. We obtain

$$\begin{aligned} &\left(i \frac{\partial}{\partial t} - A + i\lambda B^2 \right) \zeta(\phi,t) \\ &\quad = i\gamma \left\{ \sigma^2 \left(2iB + \frac{\partial}{\partial \phi} \right) \frac{\partial}{\partial \phi} + \left(iB + \frac{\partial}{\partial \phi} \right) \phi \right\} \zeta(\phi,t) \,. \end{aligned} \tag{7.12}$$

In the limit $\gamma \to 0$ and $\lambda = \gamma\sigma^2$ finite, we recover (the Fourier inverse of) Eq. (7.3) with $\hat{W}(\rho)$ from Eq. (7.7).

In order to obtain a solution of Eq. (7.12), we start with a Fourier transform with respect to ϕ. The transformed equation then reads

$$\begin{aligned} &\left(i \frac{\partial}{\partial t} - A + i\lambda B^2 \right) \hat{\zeta}(\rho,t) \\ &\quad = -i\gamma \left\{ \sigma^2(\rho - 2B)\rho + (\rho - B) \frac{\partial}{\partial \rho} \right\} \hat{\zeta}(\rho,t) \,, \end{aligned} \tag{7.13}$$

which is still a partial differential equation. Since we are interested in $\hat{\zeta}(0,t) = \langle \sigma(t) \rangle$, the obvious approach[23] would be a Taylor expansion around $\rho = 0$. This yields however a cumbersome inhomogeneous four-term recurrence relation for the Taylor coefficients. This can be avoided by the transformation[15]

$$\hat{\zeta}(\rho,t) = \hat{P}(\rho)\hat{g}(\rho,t) , \quad (7.14)$$

which defines $\hat{g}(\rho,t)$. The Fourier transform of the probability distribution is explicitly

$$\hat{P}(\rho) = \langle e^{i\rho\phi(t)}\rangle = e^{-(1/2)\sigma^2\rho^2} , \quad (7.15)$$

and in particular we have $\hat{P}(0) = 1$. The equation for $\hat{g}(\rho,t)$ becomes

$$\left(i\frac{\partial}{\partial t} - A + i\lambda B^2\right)\hat{g}(\rho,t) = -i\gamma\left\{\rho\frac{\partial}{\partial\rho} - B\left(\sigma^2\rho + \frac{\partial}{\partial\rho}\right)\right\}\hat{g}(\rho,t) , \quad (7.16)$$

and it has to be solved for

$$\hat{g}(0,t) = \langle\sigma(t)\rangle . \quad (7.17)$$

Let us define the Taylor coefficients $\pi_n(t)$ by the expansion

$$\hat{g}(\rho,t) = \sum_{n=0}^{\infty} \frac{(i\rho)^n}{n!}\pi_n(t) , \quad (7.18)$$

which can be inverted as

$$\pi_n(t) = \left\langle\sigma(t)\left(\frac{\partial}{\partial i\rho}\right)^n e^{i\rho\phi(t) - \frac{1}{2}(i\rho\sigma)^2}\right\rangle_{\rho=0} . \quad (7.19)$$

Substitution of (7.18) into (7.16) then gives the equation for the Taylor coefficients

$$\left(i\frac{\partial}{\partial t} - A + i\lambda B^2 + i\gamma n\right)\pi_n(t) = \gamma B\{n\sigma^2\pi_{n-1}(t) - \pi_{n+1}(t)\} , \quad (7.20)$$

which has to be solved for

$$\pi_0(t) = \langle\sigma(t)\rangle . \quad (7.21)$$

Equation (7.20) looks like a homogeneous three-term recurrence relation, but it will be shown below that the time derivative $\partial/\partial t$ gives rise to an inhomogeneous contribution. Notice that for $n = 0$ Eq. (7.20) reduces to a two-term relation between $\pi_0(t)$ and $\pi_1(t)$ only.

Equation (7.20) is most easily solved in the Laplace domain. If we introduce

$$\tilde{\pi}_n(\omega) = \int_{t_0}^{\infty} dt\, e^{i\omega(t-t_0)}\pi_n(t) , \quad (7.22)$$

then (7.20) attains the form

$$(\omega - A + i\lambda B^2 + i\gamma n)\tilde{\pi}(\omega) - \gamma B(n\sigma^2\tilde{\pi}_{n-1}(\omega) - \tilde{\pi}_{n+1}(\omega)) = i\pi_n(t_0) . \quad (7.23)$$

Here the initial values $\pi_n(t_0)$, for $n = 0, 1, 2,...,$ appear as inhomogeneous terms. The set $\pi_n(t_0)$ for all n represents the initial correlations of $\sigma(t)$ on $t = t_0$, and they connect the time evolution of $\langle\sigma(t)\rangle$ for $t > t_0$ to its recent past.[13] In other words, Eq. (7.23) relates the set $\pi_n(t)$ for $t > t_0$ to the initial set $\pi_n(t_0)$.

Equation (7.23) can be solved for an arbitrary initial set $\pi_n(t_0)$ by standard techniques,[24] but the solution is not transparent. In order to elucidate the structure of the solution, we assume a nonstochastic initial state $\sigma(t_0)$. From Eq. (7.1) we then find at $t = t_0$

$$\hat{\zeta}(\rho,t_0) = \hat{P}(\rho)\sigma(t_0) , \quad (7.24)$$

and from Eq. (7.14) we obtain

$$\hat{g}(\rho,t_0) = \sigma(t_0) . \quad (7.25)$$

Hence at $t = t_0$ the vector $\hat{g}(\rho,t_0)$ is independent of ρ, and therefore the expansion coefficients are simply

$$\pi_n(t_0) = \delta_{n,0}\sigma(t_0) . \quad (7.26)$$

Then only the recurrence relation for $n = 0$ is inhomogeneous, and the solution of (7.23) for $\tilde{\pi}_0(\omega) = \langle\tilde{\sigma}(\omega)\rangle$ is readily found to be

$$\langle\tilde{\sigma}(\omega)\rangle = \{i/[\omega - A + i\lambda B^2 + \tilde{K}(\omega)]\}\sigma(t_0) . \quad (7.27)$$

The effect of the finite correlation time, e.g., the deviation from the Wiener–Lévy limit, is accounted for by the operator

$$\tilde{K}(\omega) = \gamma B \cfrac{1\sigma^2}{\omega - A + i\lambda B^2 + 1i\gamma + \gamma B \cfrac{2\sigma^2}{\omega - A + i\lambda B^2 + 2i\gamma + \gamma B \cfrac{3\sigma^2}{\ddots}\gamma B}\gamma B}\gamma B , \quad (7.28)$$

which indeed vanishes for $\gamma \to 0$, λ finite. In this limit, Eq. (7.27) is the Laplace transform of Eq. (7.8).

The explicit expression (7.27) provides the exact solution for situations where the initial state is nonstochastic and for cases where the solution is independent of the initial state. As an example from quantum optics, we mention that Eq. (7.27) with $\sigma(t_0) = 1$, $A = 0$, and $B = 1$ represents the laser spectral profile. Another example pertains to the long-time behavior of the solution $\langle\sigma(t)\rangle$. If there is any damping in the system, which might be caused by the stochastic fluctuations itself, then the solution for $t \gg t_0$ will become independent of the initial state. If we indicate by $\bar{\sigma}$ the solution $\langle\sigma(t)\rangle$ for $t \to \infty$, then $\bar{\sigma}$ obviously obeys the equation

$$(A - i\lambda B^2 - \tilde{K}(0))\bar{\sigma} = 0 . \quad (7.29)$$

For the problem of atomic fluorescence in a strong laser field, this is the equation for the atomic steady-state density matrix, which determines the fluorescence yield. There, the solution $\bar{\sigma}$ of Eq. (7.29) is unique.

VIII. CONCLUSIONS

Solving the multiplicative stochastic process $\sigma(t)$ for its average is rarely feasible by analytical methods. This is mainly due to the finite correlation time of the driving process $\phi(t)$, which prohibits the factorization of the average of a product into the product of the averages. Averages of a functional of $\phi(t)$ might factorize if the process is delta correlated. For Markov processes, however, we can simulate a δ correlation by the introduction of the marginal average

$\zeta(\phi_0,t) = \langle\delta(\phi(t) - \phi_0)\sigma(t)\rangle$. The combination of the multiplication by $\delta(\phi(t) - \phi_0)$ and the Markov property of the probability distributions of $\phi(t)$ then gives rise to a factorizationlike result for the formal expression for the average. Along the same lines as in a factorization assumption, we can now derive exact equations for $\zeta(\phi_0,t)$. In this paper we have studied Eq. (1.1), where we considered the stochastics of $\phi(t)$ to be given. The equation of motion for the marginal average turned out to be Eq. (6.4). The applicability of this equation was illustrated by some examples.

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