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Squeezing in resonance fluorescence and Schrödinger's uncertainty relation

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Abstract

Resonance fluorescence can exhibit squeezing in its quadrature components. Historically, this squeezing is defined with respect to Heisenberg's uncertainty relation, but when the lower limit on the uncertainty product becomes state dependent, this concept becomes rather artificial. Schrödinger's uncertainty relation sets a higher lower bound on the uncertainty product when the two observables are correlated. In the steady state both limits are nearly equal, but for pulsed excitation they can differ considerably. It is shown that after excitation with a $\pi/2$ or π pulse, the fluorescence is never squeezed. The squeezing is optimum for a $\pi/3$ or $2\pi/3$ pulse, and is a factor of two better than for the best case in the steady state. If the inversion is zero after the pulse, then the fluctuations in the quadrature field are considerably below the Schrödinger limit, but the field is never squeezed below the Heisenberg limit.

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1. Introduction

Squeezed states are minimal uncertainty states with respect to position and momentum, but such that the variances of both operators are unequal. They are unitary transformations of the coherent states, and their properties have been known for a long time [1, 2]. Squeezed coherent states were introduced in quantum optics by Yuen and Shapiro [3] in a proposed scheme for noise reduction in optical communication systems. They also outlined how the reduction in noise in a quadrature component of such a radiation field can be observed by homodyne detection [4]. These squeezed states can be generated, in principle, in nonlinear processes like

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four-wave mixing, and much theoretical effort has been devoted to devise practical methods for squeezed-state generation [5–8]. In 1985, Slusher et al. [9] demonstrated for the first time that a squeezed state could be produced by four-wave mixing in a vapor of atomic sodium in an optical cavity, and the experiment has been reproduced and improved by others [10, 11]. Squeezed states have also been studied extensively theoretically, and most recently their phase properties were obtained [12–14]. Also generalizations have been considered, as for instance the superposition of a squeezed state and a thermal state [15, 16].

A two-state atom in a near-resonant single-mode cavity can absorb and emit photons in stimulated transitions (Jaynes–Cummings model), and it has been shown that at certain instants during the time evolution this can lead to squeezing of the radiation in the cavity [17, 18]. Also the multiphoton single-atom Jaynes–Cummings model allows for squeezing [19–21], and so does the single-mode cavity with many atoms [24–27]. In a different situation, a laser-driven two-state atom in empty space is considered, and the spontaneously-emitted resonance fluorescence is studied. It has been shown that this radiation can exhibit squeezing below the Heisenberg uncertainty limit, provided that the various optical parameters are chosen carefully [28–31]. Spectrally-resolved resonance fluorescence can also be squeezed [32, 33].

2. Quadrature field of resonance fluorescence

A two-state atom with excited state $|e\rangle$ and ground state $|g\rangle$, separated by energy $\hbar\omega_0$, is considered. The atom is irradiated by an intense laser beam, and the electric field at the position of the atom has the form

$$\boldsymbol{E}_{\mathrm{L}}(t) = E_0(t)\,\mathrm{Re}\,\boldsymbol{\varepsilon}_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{L}}t+\phi(t))}\,. \tag{2.1}$$

The amplitude $E_0(t)$ may depend on time, thereby allowing for pulsed-laser excitation, and the phase $\phi(t)$ is a stochastic process which takes into account the laser linewidth around the central frequency ω_{L}. We shall take $\phi(t)$ as the diffusive independent-increment process [34], which has the phase-diffusion model as its Gaussian limit. The laser spectral profile is then a Lorentzian, and its half-width at half maximum will be denoted by λ. The atom will emit resonance fluorescence, and the positive-frequency part of the electric field is given by [35–37].

$$E(t)^{(+)} = \gamma^{*}\,\mathrm{d}(t)^{\dagger}\,, \tag{2.2}$$

with γ an overall (complex) constant, and $d(t)^{\dagger}$ is the atomic lowering operator in the Heisenberg picture (e.g., $d^{\dagger} = |g\rangle\langle e|$).

In order to observe possible squeezing in the resonance fluorescence radiation, this field is measured by homodyne detection [4]. The driving laser is used as the local oscillator, and the mixing angle of the detector will be indicated by θ. Then the observable under measurement is the Hermitian operator [32]

$$E_{\theta}(t) = E(t)^{(+)}\,\mathrm{e}^{\mathrm{i}(\omega_{\mathrm{L}}t+\phi(t)-\theta)} + \mathrm{H.c.}\,, \tag{2.3}$$

which is a slowly-varying function of time. The output of the homodyne detector is thus proportional to the variance in this quadrature field, which will be indicated by $\mathrm{var}(E_\theta)$. In an experiment, the mixing angle θ can be varied rapidly over a large range. It should be noted that the quadrature field $E_\theta(t)$ is assumed not to vary in time with the pulse amplitude $E_0(t)$, but it only follows the time evolution of the phase of the local oscillator. This implies that for pulsed excitation the laser pulse should be split off from a CW field in order to preserve the proper phase of the quadrature field. Alternatively, one pulse could be used for both the excitation of the atom and as local oscillator. In such a scheme, the quadrature field vanishes immediately after the pulse, and this limits the observation time to the duration of the pulse. As we shall see later, the quadrature field (2.3) remains present after the pulse for a duration of the order of the lifetime of the excited state of the atom.

The variance in E_θ is defined as

$$\mathrm{var}(E_\theta) = \langle (E_\theta - \langle E_\theta \rangle)^2 \rangle = \langle E_\theta^2 \rangle - \langle E_\theta \rangle^2 . \tag{2.4}$$

We introduce the normalized variance r_θ by

$$r_\theta = \frac{\mathrm{var}(E_\theta)}{\langle E_\theta^2 \rangle} = 1 - \frac{\langle E_\theta \rangle^2}{\langle E_\theta^2 \rangle} , \tag{2.5}$$

and it follows immediately that this parameter lies in the range

$$0 \leqslant r_\theta \leqslant 1 . \tag{2.6}$$

For $r_\theta = 0$ we have $\mathrm{var}(E_\theta) = 0$, and there are no fluctuations at all in E_θ. On the other hand, for $r_\theta = 1$ the fluctuations are maximum, and such that $\langle E_\theta \rangle = 0$. This corresponds to a pure random phase of the quadrature field, a situation reminiscent of a number state.

3. Uncertainty relations

The quadrature field E_θ is an observable for all θ, and Heisenberg's uncertainty relation for quadrature components with different values of θ is

$$\mathrm{var}(E_\theta)\,\mathrm{var}(E_{\theta'}) \geqslant \tfrac{1}{4} |\langle [E_\theta, E_{\theta'}] \rangle|^2 . \tag{3.1}$$

In terms of the normalized variances this can be written as

$$r_\theta r_{\theta'} \geqslant L_\mathrm{H}^2 , \tag{3.2}$$

and here the normalized Heisenberg lower limit, L_H, is defined as

$$L_\mathrm{H} = \frac{|\langle [E_\theta, E_{\theta'}] \rangle|}{2\sqrt{\langle E_\theta^2 \rangle \langle E_{\theta'}^2 \rangle}} . \tag{3.3}$$

Then the field E_θ is said to be *squeezed* if

$$r_\theta < L_\mathrm{H} \qquad \text{(squeezing)}\,, \tag{3.4}$$

with $\theta' = \theta \pm \pi/2$ in the definition (3.3) of L_H. Therefore, squeezing is defined with respect to the Heisenberg lower limit, and with respect to the 90° out-of-phase component of the quadrature field. The squeeze function s will be defined by

$$s = r_\theta - L_\mathrm{H}\,, \tag{3.5}$$

and then the condition for squeezing becomes $s < 0$.

The correlation between the quadrature components E_θ and $E_{\theta'}$ is defined as

$$\mathrm{cor}(E_\theta, E_{\theta'}) = \tfrac{1}{2}\langle E_\theta E_{\theta'} + E_{\theta'} E_\theta\rangle - \langle E_\theta\rangle\langle E_{\theta'}\rangle\,, \tag{3.6}$$

and the correlation coefficient is

$$c = \frac{\mathrm{cor}(E_\theta, E_{\theta'})}{\sqrt{\mathrm{var}(E_\theta)\,\mathrm{var}(E_{\theta'})}}\,. \tag{3.7}$$

It can be shown that c lies in the range

$$-1 \leqslant c \leqslant 1\,. \tag{3.8}$$

Schrödinger's uncertainty relation is [38]

$$\mathrm{var}(E_\theta)\,\mathrm{var}(E_{\theta'}) \geqslant \frac{1}{4(1-c^2)}|\langle[E_\theta, E_{\theta'}]\rangle|^2\,, \tag{3.9}$$

which becomes in normalized form

$$r_\theta r_{\theta'} \geqslant L_\mathrm{S}^2\,. \tag{3.10}$$

The lower limit in Schrödinger's relation is related to the Heisenberg limit by

$$L_\mathrm{S} = \frac{1}{\sqrt{1-c^2}} L_\mathrm{H}\,, \tag{3.11}$$

and with Eq. (3.8) it then follows that

$$L_\mathrm{S} \geqslant L_\mathrm{H}\,. \tag{3.12}$$

Therefore, Schrödinger's uncertainty relation sets a higher lower bound on the product of two variances of the quadrature field, which is due to the possible correlation between the two quadrature components. For uncorrelated quadratures ($c = 0$), both uncertainty limits are equal.

4. Density operator

The state of the atom is represented by a density operator $\rho(t)$, which has a 2×2 matrix representation with respect to the atomic states. This operator is a stochastic process due to the random phase $\phi(t)$ of the driving laser. We shall indicate by $\prec \dots \succ$ an average over the phase fluctuations. Then the populations n_e and n_g of the atomic levels are related to $\rho(t)$ by

$$n_e(t) = \prec\langle e|\rho(t)|e\rangle\succ , \qquad n_g(t) = \prec\langle g|\rho(t)|g\rangle\succ . \tag{4.1}$$

The coherence between the levels oscillates with the optical frequency ω_L and varies stochastically with $\phi(t)$. The quantity of interest is the slowly-varying and averaged coherence $\sigma_{eg}(t)$, defined by

$$\sigma_{eg}(t) = \prec e^{i(\omega_L t + \phi(t))}\langle e|\rho(t)|g\rangle\succ . \tag{4.2}$$

The notation $\langle \dots \rangle$, like in the definition (2.4) of the variance, implies both a quantum average and an average over $\phi(t)$.

Eq. (2.3) gives the Heisenberg operator for the quadrature field $E_\theta(t)$, and with Eq. (2.2) this can be expressed in terms of the atomic raising and lowering operators. Transforming to the Schrödinger picture and taking the average over the stochastic fluctuations then yields

$$\langle E_\theta(t)\rangle = e^{-i\theta}\gamma^* \sigma_{eg}(t) + \text{c.c} . \tag{4.3}$$

With the property

$$d(t)^2 = \{d(t)^\dagger\}^2 = 0 , \tag{4.4}$$

we obtain for the square of the operator $E_\theta(t)$

$$E_\theta(t)^2 = \gamma_0^2\{d(t)^\dagger d(t) + d(t) d(t)^\dagger\} , \tag{4.5}$$

where we have set $\gamma_0 = |\gamma|$. But the term in curly brackets is just the unit operator, so that $E_\theta(t)^2 = \gamma_0^2$, and therefore

$$\langle E_\theta(t)^2\rangle = \gamma_0^2 . \tag{4.6}$$

The normalized variance r_θ then becomes

$$r_\theta = 1 - 4|\sigma_{eg}|^2 \cos^2\delta , \tag{4.7}$$

where we introduced the phase angle δ:

$$\delta = \theta + \arg(\gamma) - \arg(\sigma_{eg}) . \tag{4.8}$$

This angle δ is the homodyne mixing angle θ, shifted by a fixed amount. During an experiment, the mixing angle is swept quickly over a range of at least π, and therefore the shifting of θ is irrelevant. Then Eq. (4.7) shows that the fluctuations in the quadrature field are determined entirely by the atomic coherence and δ. Notice that σ_{eg} will, in general, depend on time. Consequently, the variance r_θ depends on time, and also the shift of angle θ. This time dependence, however, is merely parametric because angle θ is varied much more rapidly than the time evolution of the coherence.

5. Uncertainty limits

The Heisenberg lower limit L_{H} from Eq. (3.3) is determined by the equal-time commutator of the quadrature field at different mixing angles. With Eqs. (2.2) and (2.3) we find

$$[E_\theta(t), E_{\theta'}(t)] = 2\mathrm{i}\gamma_0^2\{P_g(t) - P_e(t)\}\sin(\theta' - \theta)\,, \tag{5.1}$$

in terms of the projection operators $P_g = |g\rangle\langle g|$ and $P_e = |e\rangle\langle e|$ onto the atomic levels, and in the Heisenberg picture. The difference between the phase angles will be taken as $\pi/2$, which means that we compare the fluctuations in the quadrature field E_θ with the fluctuations in its 90° out-of-phase component. Taking the expectation value and absolute value of Eq. (5.1) then yields

$$|\langle[E_\theta(t), E_{\theta'}(t)]\rangle| = 2\gamma_0^2|n_g(t) - n_e(t)|\,, \tag{5.2}$$

and with Eq. (4.6) we then obtain

$$L_{\mathrm{H}} = |n_e(t) - n_g(t)|\,. \tag{5.3}$$

Hence the Heisenberg lower limit is simply equal to the absolute value of the atomic inversion. Since the population of an atomic level can not exceed unity, the value of L_{H} is limited to

$$0 \leqslant L_{\mathrm{H}} \leqslant 1\,. \tag{5.4}$$

The squeeze function s becomes

$$s = 1 - 4|\sigma_{eg}|^2\cos^2\delta - |n_e(t) - n_g(t)|\,, \tag{5.5}$$

which is determined entirely by the matrix elements of the atomic density operator and by the mixing angle δ. Its value are in the range

$$-1 \leqslant s \leqslant 1\,, \tag{5.6}$$

and squeezing occurs for $s < 0$.

The correlation between two 90° out-of-phase quadrature components is found to be

$$\operatorname{cor}(E_\theta, E_{\theta'}) = 2\gamma_0^2|\sigma_{eg}|^2\sin(2\delta)\,, \tag{5.7}$$

which gives for the correlation coefficient

$$c = \frac{2|\sigma_{eg}|^2\sin(2\delta)}{\sqrt{\{1 - 4|\sigma_{eg}|^2\cos^2\delta\}\{1 - 4|\sigma_{eg}|^2\sin^2\delta\}}}\,. \tag{5.8}$$

The relation between L_{S} and L_H then becomes

$$L_{\mathrm{S}} = L_{\mathrm{H}}\sqrt{1 + \frac{4|\sigma_{eg}|^4}{1 - 4|\sigma_{eg}|^2}\sin^2(2\delta)}\,. \tag{5.9}$$

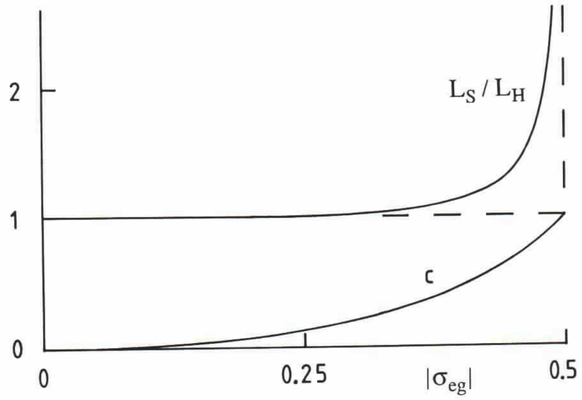

Fig. 1. Illustration of the ratio L_S/L_H and the correlation coefficient c for $\delta = \pi/4$ or $3\pi/4$, as a function of the absolute value of the coherence. For small values of $|\sigma_{eg}|$, both uncertainty limits are nearly equal, and c vanishes. For larger coherences, the ratio L_S/L_H approaches infinity and the correlation coefficient tends to unity.

This shows that the ratio L_S/L_H is determined by the coherence and the angle δ. The dependence on δ is periodic with $\pi/2$, and the parameter r_θ is periodic with π. Therefore, we shall restrict the values of δ to $0 \leqslant \delta \leqslant \pi$. Notice that the Heisenberg limit is independent of δ. For $\delta = 0$, $\pi/2$ and π we find $L_S = L_H$, and from (5.7) we see that this is due to the fact that the two quadrature components are uncorrelated for these values of δ. The correlation is maximum for $\delta = \pi/4$ and $3\pi/4$, and for these angles the Schrödinger limit deviates the most from the Heisenberg limit. For these angles the relation becomes

$$L_S = L_H \frac{1 - 2|\sigma_{eg}|^2}{\sqrt{1 - 4|\sigma_{eg}|^2}} . \tag{5.10}$$

Another interesting point to notice is that for a two-level system the absolute value of the coherence is limited to the range $0 \leqslant |\sigma_{eg}| \leqslant \frac{1}{2}$. From Eq. (5.9) we then see that for $|\sigma_{eg}| = 0$ we have $L_S = L_H$, where for $|\sigma_{eg}| \to \frac{1}{2}$ the ratio L_S/L_H approaches infinity. The behavior of L_S/L_H as a function of the coherence is illustrated in Fig. 1. It should be noted that $L_S/L_H \to \infty$ does not necessarily imply that L_S goes to infinity, as we shall see later.

6. Equation of motion

The time evolution of the quadrature field, its fluctuations, and its uncertainty limits are determined by the time evolution of the atomic density operator, as shown above. It is convenient to represent the (transformed and averaged) atomic density operator matrix elements by the three components of the Bloch vector, defined by [39]

$$\begin{cases} u = 2\,\mathrm{Re}\,\sigma_{eg}, \\ v = -2\,\mathrm{Im}\,\sigma_{eg}, \\ w = n_e - n_g. \end{cases} \tag{6.1}$$

The equations of motion for these components are [40,41]

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = -\eta u + \Delta v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\eta v - \Delta u + \Omega w, \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = -A(w+1) - \Omega v, \end{cases} \tag{6.2}$$

with A the Einstein coefficient for spontaneous decay, $\eta = \lambda + A/2$ the laser linewidth parameter, $\Delta = \omega_{\mathrm{L}} - \omega_0$ the detuning, and Ω the Rabi frequency, defined by

$$\Omega(t) = \frac{1}{\hbar} E_0(t) \langle \mathrm{e} | \boldsymbol{\mu} \cdot \boldsymbol{\varepsilon}_{\mathrm{L}} | g \rangle , \tag{6.3}$$

and we shall assume $\Omega(t) \geqslant 0$. The vector $\boldsymbol{\mu}$ is the atomic dipole operator.

The damping in the time evolution is brought about by spontaneous decay and the laser linewidth. From Eqs. (6.2) we immediately find

$$\frac{\mathrm{d}}{\mathrm{d}t}(u^2 + v^2 + w^2) = -2\eta(u^2 + v^2) - 2Aw(w+1) , \tag{6.4}$$

showing that without damping the quantity $u^2 + v^2 + w^2$ is conserved. For a pure state at $t = 0$ we have [39] $u^2 + v^2 + w^2 = 1$ at $t = 0$, and without damping this remains unity for all times.

The Heisenberg lower limit L_{H} and the variance r_θ are in terms of u, v, and w,

$$L_{\mathrm{H}} = |w| , \tag{6.5}$$

$$r_\theta = 1 - (u^2 + v^2)\cos^2\delta , \tag{6.6}$$

and then $s = r_\theta - L_{\mathrm{H}}$.

7. Steady state

For CW excitation of the atom, $E_0(t)$ and $\Omega(t)$ are constant, and u, v, and w will reach a steady state in a time of the order of $1/A$. In the long-time limit the time derivatives on the left-hand sides of Eqs. (6.2) vanish, and the set is easily solved. We obtain:

$$u = -\frac{\Omega A \Delta}{\Omega^2 \eta + A(\Delta^2 + \eta^2)} , \tag{7.1}$$

$$v = -\frac{\Omega A \eta}{\Omega^2 \eta + A(\Delta^2 + \eta^2)}, \tag{7.2}$$

$$w = -\frac{A(\Delta^2 + \eta^2)}{\Omega^2 \eta + A(\Delta^2 + \eta^2)}. \tag{7.3}$$

The squeeze function s becomes in the steady state

$$s = \Omega^2 \frac{\Omega^2 \eta^2 + A(\Delta^2 + \eta^2)(\eta - A\cos^2\delta)}{\{\Omega^2 \eta + A(\Delta^2 + \eta^2)\}^2}. \tag{7.4}$$

As a function of δ, this is minimum for $\delta = 0$ and $\delta = \pi$, and s is negative for these angles under the condition

$$\Omega^2 \eta^2 \langle A(\Delta^2 + \eta^2)(A - \eta) \qquad \text{(squeezing)}, \tag{7.5}$$

as has been found before [29]. Squeezing can only occur for $A - \eta > 0$, which is $\lambda < A/2$.

Given the laser linewidth λ (smaller than $A/2$) and detuning Δ, the laser power (proportional to Ω^2) can be chosen such as to minimize the value of s (for $\delta = 0$ and π). This optimum laser power is

$$\Omega^2 = \frac{A(A - \eta)}{\eta(A + \eta)}(\Delta^2 + \eta^2), \tag{7.6}$$

and for this value of Ω^2 the squeeze function becomes

$$s = -\frac{(A - \eta)^2}{4A\eta} < 0. \tag{7.7}$$

Consequently, for the optimum laser power the quadrature field of resonance fluorescence is always squeezed (for $\delta = 0$ and π). The ultimate minimum is $s = -1/8$, which is attained for $\lambda = 0$ ($\eta = A/2$).

We introduce the new parameters

$$\xi = \frac{\Omega^2}{\Delta^2 + \eta^2}, \qquad \hat{\eta} = \frac{\eta}{A}, \tag{7.8}$$

in terms of which r_θ, L_{H}, and s become

$$r_\theta = 1 - \frac{\xi}{(\xi\hat{\eta} + 1)^2}\cos^2\delta, \tag{7.9}$$

$$L_{\mathrm{H}} = \frac{1}{\xi\hat{\eta} + 1}, \tag{7.10}$$

$$s = \xi \frac{\hat{\eta}(\xi\hat{\eta} + 1) - \cos^2\delta}{(\xi\hat{\eta} + 1)^2}. \tag{7.11}$$

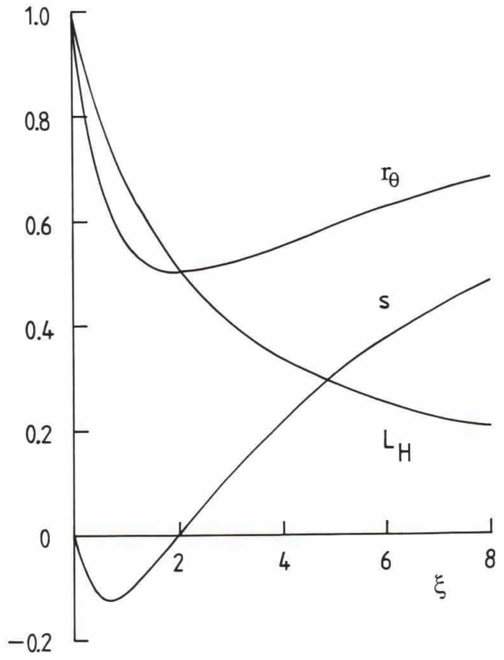

Fig. 2. Plots of r_θ, L_{H}, and s as a function of the parameter ξ for $\hat{\eta}=\frac{1}{2}$ and $\delta=0$ or π. The minimum of $s=-\frac{1}{8}$ occurs at $\xi=\frac{2}{3}$, and the variance has its minimum of $r_\theta=\frac{1}{2}$ at $\xi=2$.

In terms of these parameters, the minimum values of $s=-1/8$ occurs for $\delta=0$ or π, $\hat{\eta}=\frac{1}{2}$, and $\xi=\frac{2}{3}$. Fig. 2 shows r_θ, L_{H}, and s as a function of ξ, for $\delta=0$ or π and $\hat{\eta}=\frac{1}{2}$. It should be noted that the squeeze function is minimum for $\xi=\frac{2}{3}$, but that the variance has its minimum of $r_\theta=\frac{1}{2}$ for $\xi=2$.

8. Schrödinger limit

In the steady state, the Schrödinger limit L_{S} from Eq. (5.9) is related to the Heisenberg limit L_{H} as

$$L_{\mathrm{S}}=L_{\mathrm{H}}\sqrt{1+\left(\frac{\xi/2}{\xi\hat{\eta}+1}\right)^2\frac{\sin^2(2\delta)}{(\xi\hat{\eta}+1)^2-\xi}}, \tag{8.1}$$

with L_{H} given by Eq. (7.10). Since for $\delta=0$, $\pi/2$ and π both uncertainty limits are equal, we consider here the dependence on the mixing angle δ.

Given the detuning and the laser linewidth, the laser power that minimizes the squeeze function s is

$$\xi=\frac{1-\hat{\eta}}{\hat{\eta}(1+\hat{\eta})}, \tag{8.2}$$

according to Eq. (7.6). For this value of ξ, the variance of the fluctuations becomes

$$r_\theta = 1 - \frac{1}{4\hat{\eta}}(1-\hat{\eta}^2)\cos^2\delta\,, \tag{8.3}$$

and the uncertainty limits are

$$L_\mathrm{H} = \tfrac{1}{2}(1+\hat{\eta})\,, \tag{8.4}$$

$$L_\mathrm{S} = L_\mathrm{H}\sqrt{1+\frac{1}{16\hat{\eta}}\frac{(1-\hat{\eta}^2)^2\sin^2(2\delta)}{\hat{\eta}^2+4\hat{\eta}-1}}\,. \tag{8.5}$$

As a function of $\hat{\eta}$, the squeezing is optimum for $\hat{\eta}=\frac{1}{2}$. For this limit of zero laser linewidth, we have $\xi=\frac{2}{3}$, $L_\mathrm{H}=\frac{3}{4}$, and

$$r_\theta = 1 - \tfrac{3}{8}\cos^2\delta\,, \tag{8.6}$$

$$L_\mathrm{S} = \tfrac{3}{4}\sqrt{1+\tfrac{9}{160}\sin^2(2\delta)}\,. \tag{8.7}$$

Fig. 3 shows r_θ, L_H and L_S as a function of δ. The maximum difference between L_H and L_S is 0.02, which occurs for $\delta=\pi/4$ and $3\pi/4$. For $\delta=\pi/2$ we have $r_\theta=1$, and this corresponds to a pure random phase in the quadrature field. For $\delta=0$ or π we have the minimum value of $r_\theta=5/8$, corresponding to the minimum value $s=-1/8$ of the squeeze function.

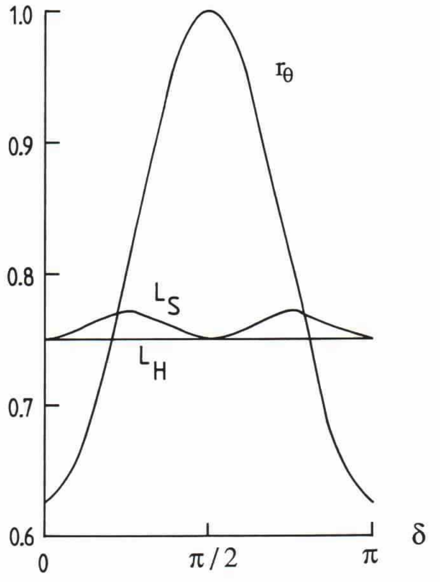

Fig. 3. Graphs of r_θ, L_H and L_S for steady-state resonance fluorescence, as a function of δ, for $\hat{\eta}=\frac{1}{2}$ and the value of ξ that minimizes s. The Heisenberg limit is $\frac{3}{4}$, and the Schrödinger limit is slightly higher for $\delta \neq 0$, $\pi/2$ or π.

9. Pulsed excitation

Although squeezing of the resonance fluorescence can be achieved in the steady state, it requires a delicate combination of optical parameters, and the squeezing is not very pronounced. As an alternative, we consider atomic excitation with a short laser pulse, and the possibility for squeezing in the fluorescence which is emitted after the pulse. If the pulse is short compared to the lifetime of the excited state and compared to the inverse laser linewidth due to phase fluctuations, then we can set $A = \lambda = 0$ (and thereby $\eta = 0$) in Eqs. (6.2), and for simplicity we shall assume perfect resonance. Then the equations of motion during the pulse become

$$\frac{du}{dt} = 0, \qquad \frac{dv}{dt} = \Omega w, \qquad \frac{dw}{dt} = -\Omega v\,. \tag{9.1}$$

We take $t = 0$ as the instant just before the pulse, and we shall assume that the atom is then in the ground state. This gives as initial conditions $u(0) = v(0) = 0$ and $w(0) = -1$. With Eq. (6.4) we then obtain the relation

$$u^2 + v^2 + w^2 = 1\,, \tag{9.2}$$

which holds for all t during the pulse.

From Eqs. (6.1) and (9.2) we have

$$|\sigma_{eg}|^2 = \tfrac{1}{4}(u^2 + v^2) = \tfrac{1}{4}(1 - w^2)\,, \tag{9.3}$$

and this allows us to express the variance r_θ, as given by Eq. (4.7), in terms of the inversion

$$r_\theta = 1 - (1 - w^2)\cos^2\delta\,. \tag{9.4}$$

The Heisenberg limit from Eq. (5.3) becomes

$$L_{\mathrm{H}} = |w|\,, \tag{9.5}$$

and with Eq. (5.9) we can also express L_{S} in terms of the inversion. This yields

$$L_{\mathrm{S}} = \tfrac{1}{2}\sqrt{4w^2 + (1 - w^2)^2\sin^2(2\delta)}\,. \tag{9.6}$$

The difference between L_{S} and L_{H} is maximum for $\delta = \pi/4$ and $3\pi/4$, for which the Schrödinger limit equals $L_{\mathrm{S}} = \frac{1}{2}(1 + w^2)$, and this limit has its maximum of $L_{\mathrm{S}} = 1$ at $|w| = 1$ (atom in $|e\rangle$ or in $|g\rangle$). From Eq. (5.9), or as shown in Fig. 1, the ratio $L_{\mathrm{S}}/L_{\mathrm{H}}$ approaches infinity when the coherence approaches $\frac{1}{2}$. This corresponds, however, to $w \to 0$, and in this limit the Schrödinger uncertainty becomes $L_{\mathrm{S}} = \frac{1}{2}|\sin(2\delta)|$, which is finite. In this limit, $L_{\mathrm{H}} \to 0$, and therefore we have $L_{\mathrm{S}}/L_{\mathrm{H}} \to \infty$. The squeeze function is

$$s = 1 - (1 - w^2)\cos^2\delta - |w|\,. \tag{9.7}$$

As a function of δ this has a minimum of $s = w^2 - |w|$ for $\delta = 0$ or π, and then the minimum of s becomes $s = -\frac{1}{4}$, for $|w| = \frac{1}{2}$. In terms of the atomic populations this means $n_e = \frac{1}{4}$, $n_g = \frac{3}{4}$ $(w = -\frac{1}{2})$ or $n_e = \frac{3}{4}$, $n_g = \frac{1}{4}$ $(w = \frac{1}{2})$. This minimum of s is a factor of two lower than the best possible value of $s = -\frac{1}{8}$ for CW excitation.

The solution of the set (9.1), with the atom initially in the ground state, is

$$u(t) = 0, \qquad v(t) = -\sin \Phi(t), \qquad w(t) = -\cos \Phi(t)\,, \tag{9.8}$$

where

$$\Phi(t) = \int_0^t \mathrm{d}t'\, \Omega(t')\,. \tag{9.9}$$

If we let t be the time at the end of the pulse, then Φ is the pulse area.

In order to obtain maximum squeezing just after the pulse, we must take the pulse area to be $\Phi = \pi/3$ (for $w = -\frac{1}{2}$) or $\Phi = 2\pi/3$ (for $w = \frac{1}{2}$). After excitation with such a pulse, the fluctuations are

$$r_\theta = 1 - \tfrac{3}{4}\cos^2 \delta\,, \tag{9.10}$$

the Heisenberg limit is $L_{\mathrm{H}} = \frac{1}{2}$, and the squeeze function becomes

$$s = \tfrac{1}{8}(1 - 3\cos(2\delta))\,, \tag{9.11}$$

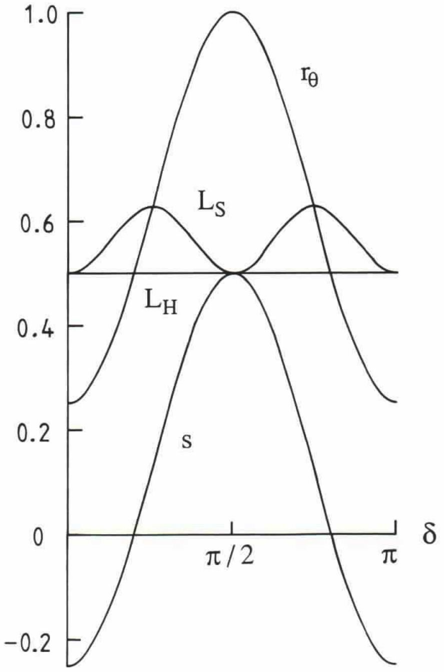

Fig. 4. Plots of r_θ, L_{H}, s and L_{S} for pulsed-laser excitation with $|w| = \frac{1}{2}$ after the pulse. The curves are parameter free.

representing the best possible squeezing with respect to the Heisenberg limit. Furthermore, we have for the Schrödinger limit in this case

$$L_S = \tfrac{1}{2}\sqrt{1 + \tfrac{9}{16}\sin^2(2\delta)}\,. \tag{9.12}$$

The maximum deviation from the Heisenberg limit is $L_S = \frac{5}{4}L_H$, which occurs for $\delta = \pi/4$ and $3\pi/4$. The fluctuations are squeezed for $0 < \delta < 35.3°$ and $144.7° < \delta < 180°$ (in the range from 0° to 180°). The various quantities for $|w| = \frac{1}{2}$ are shown in Fig. 4.

Another interesting situation is excitation with a $\Phi = \pi/2$ pulse, for which $|w| = 0$ after the pulse, corresponding to equal populations of both levels. This yields $|\sigma_{eg}| = \frac{1}{2}$, which is the maximum possible value of coherence in a two-state system. From Eq. (9.5) we then find $L_H = 0$, and consequently squeezing below the Heisenberg limit can never occur. The variance in the fluctuations is $r_\theta = \sin^2\delta$, which gives $r_\theta = 0$ for $\delta = 0$ and π, corresponding to no fluctuations at all. The Schrödinger limit becomes $L_S = \frac{1}{2}|\sin(2\delta)|$. For $0 < \delta < \pi/4$ and $3\pi/4 < \delta < \pi$ the fluctuations drop below the Schrödinger limit, and $r_\theta - L_S$ has a minimum of $\frac{1}{2}(1 - \sqrt{2}) = -0.207$ at $\delta = \pi/8$ and $7\pi/8$. This behavior is shown in Fig. 5.

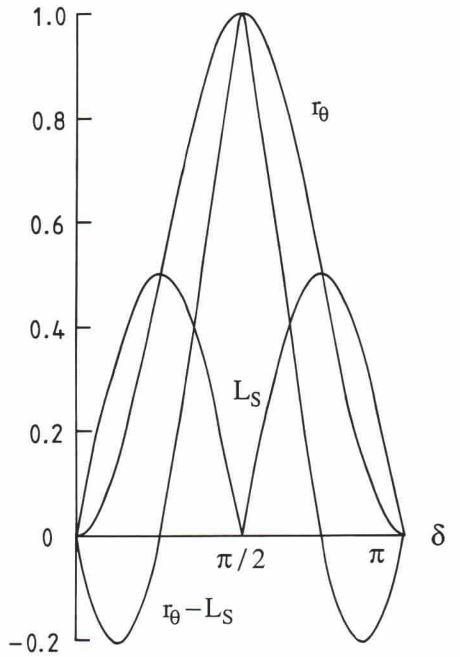

Fig. 5. Graphs of r_θ, L_S and $r_\theta - L_S$ as a function of δ, for pushed excitation with $|w| = 0$ after the pulse. The Heisenberg limit here is zero. For certain values of δ, the variance of the fluctuations is well below the Schrödinger limit.

Another interesting observation is that after excitation with a π pulse, which gives $w = 1$ and leaves the atom in the excited state without any coherence, we have $r_\theta = L_{\mathrm{H}} = L_{\mathrm{S}} = 1$, and squeezing never occurs.

Finally, for $\delta = 0$ or π after excitation with any pulse we have $L_{\mathrm{H}} = L_{\mathrm{S}} = |w|, r_\theta = w^2$, and $s = w^2 - |w|$. This shows that s is negative after any pulse, except for pulses which give $|w| = 0$ or 1. Hence squeezing can be obtained easily with pulsed excitation, and the optimum pulse gives $|w| = \frac{1}{2}$, with $s = -\frac{1}{4}$.

10. After the pulse

After excitation with a pulse, the atom decays to its ground state. The time evolution of the quadrature field during the decay is governed by the equations of motion (6.2), which reduce to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\tfrac{1}{2}Au, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -\tfrac{1}{2}Av, \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = -A(w+1)\,, \tag{10.1}$$

for $\Delta = \lambda = 0$. If the atom starts out in its ground state before the pulse, then immediately after the pulse the values of u, v, and w are $u_{\mathrm{a}} = 0$, $v_{\mathrm{a}} = -\sin\Phi$, $w_{\mathrm{a}} = -\cos\Phi$, according to Eqs. (9.8), and where Φ is the pulse area. The solution of Eqs. (10.1) is

$$u(t) = 0, \qquad v(t) = v_{\mathrm{a}}\,\mathrm{e}^{-At/2}, \qquad w(t) = (1 + w_{\mathrm{a}})\,\mathrm{e}^{-At} - 1\,, \tag{10.2}$$

where we have redefined $t = 0$ as the time just after the pulse.

After the pulse we have $u_{\mathrm{a}}^2 + v_{\mathrm{a}}^2 + w_{\mathrm{a}}^2 = 1$, but during the relaxation the quantity $u^2 + v^2 + w^2$ is not conserved and not equal to unity. For the coherence we obtain

$$|\sigma_{eg}|^2 = \tfrac{1}{4}(1 - w_{\mathrm{a}}^2)\,\mathrm{e}^{-At}\,, \tag{10.3}$$

which gives r_θ with Eq. (4.7), and the Heisenberg limit is $L_{\mathrm{H}} = |w|$. For $\delta = 0$ or π the squeeze function then becomes

$$s = \begin{cases} w_{\mathrm{a}}(1 + w_{\mathrm{a}})\,\mathrm{e}^{-At} & \text{for } \begin{cases} w_{\mathrm{a}} < 0, \\ w_{\mathrm{a}} > 0, \quad At > \ln(1 + w_{\mathrm{a}}), \end{cases} \\ 2 - (1 + w_{\mathrm{a}})(2 - w_{\mathrm{a}})\,\mathrm{e}^{-At} & \text{for } w_{\mathrm{a}} > 0, \quad At < \ln(1 + w_{\mathrm{a}}) \end{cases} . \tag{10.4}$$

For a negative inversion after the pulse, s remains negative for all times. For a positive inversion after the pulse, s is negative up to $At = \ln(1 + w_{\mathrm{a}})(1 - w_{\mathrm{a}}/2)$, and then positive for larger times. For $At = \ln(1 + w_{\mathrm{a}})$, the squeeze function has its maximum of $s = w_{\mathrm{a}} > 0$. This behavior is illustrated in Fig. 6 for $w_{\mathrm{a}} = \pm\frac{1}{2}$.

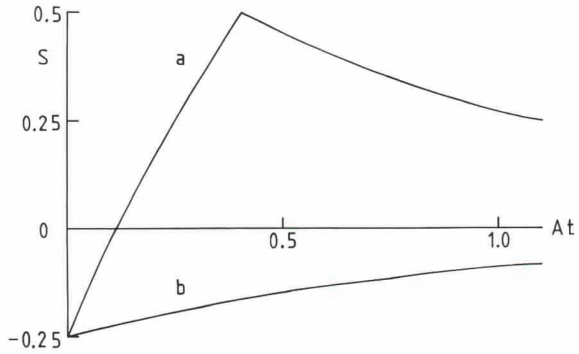

Fig. 6. Time evolution of the squeeze function s (shown as a function of At) after pulsed excitation with inversion $|w_a| = \frac{1}{2}$ just after the pulse. Curve a corresponds to $w_a = \frac{1}{2}$, and curve b represents $w_a = -\frac{1}{2}$. If the inversion is negative after the pulse, the quadrature field remains squeezed for all later times.

11. Conclusions

Fluctuations in the quadrature field of resonance fluorescence from a two-state atom can be squeezed below the Heisenberg uncertainty limit. In the steady state, the squeeze function $s = r_\theta - L_H$ has a minimum of $-\frac{1}{8}$ for zero laser linewidth and, given the detuning, a Rabi frequency equal to

$$\Omega = \sqrt{\tfrac{2}{3}(\Delta^2 + \tfrac{1}{4}A^2)}\,, \tag{11.1}$$

as follows from Eq. (7.6) with $\eta = A/2$. For pulsed excitation of the atom, the minimum value of s immediately after the pulse is $s = -\frac{1}{4}$. This can be accomplished with either a $\pi/3$ or a $2\pi/3$ pulse, incident upon the atom in the ground state. After such a pulse, the inversion is $w = -\frac{1}{2}$ and $w = \frac{1}{2}$, respectively. When the inversion equals $w = 0$ or $w = \pm 1$ after a pulsed excitation, the fluorescence is never squeezed.

The Heisenberg uncertainty limit is commonly used as the reference level for the definition of squeezing in observables. However, the Schrödinger uncertainty limit sets a higher bound on quantum fluctuations, and this limit could alternatively be used to define squeezing. Especially in the present problem, where both uncertainty limits are state dependent, the choice of reference level becomes rather artificial. It has been shown that for resonance fluorescence both limits are nearly equal in the steady state. For pulsed excitation, on the other hand, both limits can differ significantly. For instance, after excitation with a $\pi/2$ pulse from the ground state, the quadrature field can be squeezed well below the Schrödinger limit, but never below the Heisenberg limit.

Acknowledgements

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References

[1] D. Stoler, Phys. Rev. D 1 (1970) 3217.
[2] D. Stoler, Phys. Rev. D 4 (1971) 1925.
[3] H.P. Yuen and J.H. Shapiro, IEEE Trans. Inf. Theory IT-24 (1978) 657.
[4] J.H. Shapiro, H.P. Yuen and J.A.M. Mata, IEEE Trans. Inf. Theory IT-25 (1979) 179.
[5] H.P. Yuen, Phys. Rev. A 13 (1976) 2226.
[6] H.P. Yuen and J.H. Shapiro, Opt. Lett. 4 (1979) 334.
[7] M.D. Reid and D.F. Walls, Phys. Rev. A 31 (1985) 1622.
[8] J. Bergou, C. Benkert, L. Davidovich, M.O. Scully, S.Y. Zhu and M.S. Zubairy, Phys. Rev. A 42 (1990) 5544.
[9] R.E. Slusher, R.W. Hollberg, B. Yurke, J.C. Mertz and J.F. Valley, Phys. Rev. Lett. 55 (1985) 2409.
[10] L.A. Wu, H.J. Kimble, J.L. Hall and H. Wu, Phys. Rev. Lett. 57 (1986) 2520.
[11] D.M. Hope, D.E. McClelland, H.A. Bachor and A.J. Stevenson, Appl. Phys. B 55 (1992) 210.
[12] H.Y. Fan and H.R. Zaidi, Opt. Commun. 68 (1988) 143.
[13] J.A. Vaccaro and D.T. Pegg, Opt. Commun. 70 (1989) 529.
[14] N. Grønbech-Jensen, P.L. Christiansen and P.S. Ramanujam, J. Opt. Soc. Am. B 6 (1989) 2423.
[15] P. Marian and T.A. Marian, Phys. Rev. A 47 (1993) 4474.
[16] P. Marian and T.A. Marian, Phys. Rev. A 47 (1993) 4487.
[17] P. Meystre and M.S. Zubairy, Phys. Lett. A 89 (1982) 390.
[18] J.R. Kuklin'ski and J.L. Madajczyk, Phys. Rev. A 37 (1988) 3175.
[19] F.L. Kien, M. Kozierowski and T. Quang, Phys. Rev. A 38 (1988) 263.
[20] X.S. Li, D.L. Lin and T.F. George, Phys. Rev. A 40 (1989) 2504.
[21] Y.J. Xia and G.C. Guo, J. Opt. Soc. Am. B 7 (1990) 2113.
[22] P. Zhou and J.S. Peng, Phys. Rev. A 44 (1991) 3331.
[23] T. Nasreen and M.S.K. Razmi, Opt. Commun. 88 (1992) 101.
[24] A. Heidmann, J.M. Raimond and S. Reynaud, Phys. Rev. Lett. 54 (1985) 326.
[25] W. Vogel and D.G. Welsch, Phys. Rev. Lett. 54 (1985) 1802.
[26] F.L. Li, X.S. Li, D.L. Lin and T.F. George, Phys. Rev. A 41 (1990) 2712.
[27] P.L. Knight and T. Quang, Phys. Rev. A 41 (1990) 6255.
[28] D.F. Walls and P. Zoller, Phys. Rev. Lett. 47 (1981) 709.
[29] H.F. Arnoldus and G. Nienhuis, Optica Acta 30 (1983) 1573.
[30] J.M. Courty and S. Reynaud, Opt. Commun. 72 (1989) 93.
[31] G.Y. Kryuchkyan, Quantum Opt. 3 (1991) 209.
[32] M.J. Collett, D.F. Walls and P. Zoller, Opt. Commun. 52 (1984) 145.
[33] V. Buzek and T. Quang, J. Mod. Opt. 38 (1991) 1559.
[34] N.G. van Kampen, Stochastic Processes in Physics and Chemistry (North-Holland, Amsterdam, 1981), p. 94.
[35] B.R. Mollow, Phys. Rev. 188 (1969) 1969.
[36] H.J. Carmichael and D.F. Walls, J. Phys. B 9 (1976) 1199.
[37] H.J. Kimble and L. Mandel, Phys. Rev. A 13 (1976) 2123.
[38] V.V. Dodonov, E.V. Kurmyshev and V.I. Man'ko, Phys. Lett. A 79 (1980) 150.
[39] L. Allen and J.H. Eberly, Optical Resonance and Two-Level Atoms (Dover Publ., New York, 1987), p. 45.
[40] G.S. Agarwal, Phys. Rev. Lett. 37 (1976) 1383.
[41] H.F. Arnoldus and G. Nienhuis, J. Phys. B 16 (1983) 2325.