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Temporal correlations between photon detections from damped single-mode radiation

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Abstract

Radiation in a single-mode cavity will evolve from some given initial state to thermal equilibrium with the cavity mirrors. The temporal, time-dependent, photon intensity correlation functions (joint detection probabilities) have been obtained to all orders. The result is a sum over combinatorial products connecting the detection times.

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1. Introduction

When radiation in a single-mode cavity is prepared in an initial state, described by the density operator $\rho(0)$, then the interaction with the mirrors will lead to relaxation of the radiation to the unique thermal equilibrium state, which is determined by the temperature only. The transient state density operator $\rho(t)$ obeys the Liouville equation [1]

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = L\rho, \tag{1}$$

with the Liouvillian L given by

$$\begin{aligned} L\rho = \omega_c[a^\dagger a,\rho] &- \frac{1}{2}\mathrm{i}Kn_{eq}(aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a) \\ &- \frac{1}{2}\mathrm{i}K(n_{eq}+1)(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger). \end{aligned} \tag{2}$$

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Here, ω_c is the cavity frequency, K is the cavity damping rate, and n_{eq} is the number of photons in thermal equilibrium. This parameter represents the temperature T according to

$$n_{eq} = \frac{1}{\exp(\hbar\omega_c/kT) - 1}, \tag{3}$$

with k Boltzmann's constant. The radiation in the cavity can be observed by means of counting photons with a photon absorption counter. Properties of the radiation, and especially quantum dynamical features, are then revealed in the statistical distribution in time of the photon counts. In particular, the temporal correlations between photons detected from the cavity radiation are directly amenable to observation. We consider the joint probability distribution $I_k(t_1,t_2,\ldots,t_k)$ for the detection of k photons at discrete times. To be specific, $I_k(t_1,t_2,\ldots,t_k)\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_k$ equals the probability for the detection of a photon in $[t_1,t_1+\mathrm{d}t_1]$, a photon in $[t_2,t_2+\mathrm{d}t_2]$, and $\ldots$ and the detection of a photon in $[t_k,t_k+\mathrm{d}t_k]$, irrespective of detections at other times. For classical radiation, these correlation functions factor as $I_k(t_1,t_2,\ldots,t_k) = \langle I_1(t_1)I_1(t_2)\ldots I_1(t_k)\rangle$, with $\langle\ldots\rangle$ an average over stochastic fluctuations [2]. For quantum radiation, however, these photon correlations are proportional to the field intensity correlations as follows [3,4]:

$$I_k(t_1,\ldots,t_k) = \zeta^k\langle a^\dagger(t_1)\ldots a^\dagger(t_k)a(t_k)\ldots a(t_1)\rangle, \tag{4}$$

where ζ is an overall detection efficiency parameter, the time order is $t_k > \ldots > t_2 > t_1$, and the time dependence of the creation and annihilation operators signifies the Heisenberg picture representation. These correlation functions have been studied by several authors [5,6], who express $I_k(t_1,t_2,\ldots,t_k)$ as a multiple integral over quasi-probability distributions (P-, Q- or Wigner-representations), but it is not clear how these expressions can be used to evaluate the correlation functions for any order k. Here we present a different approach, leading to an explicit result for $I_k(t_1,t_2,\ldots,t_k)$.

2. Photon field statistics

The probability $p_n(t)$ for finding n photons in the cavity at time t is equal to the diagonal matrix element $\langle n|\rho(t)|n\rangle$ of the density operator. From the equation of motion (1) we then derive

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -Kn_{eq}\{(n+1)p_n - np_{n-1}\} - K(n_{eq}+1)\{np_n - (n+1)p_{n+1}\}, \tag{5}$$

with $p_{-1} \equiv 0$. This equation can be solved in many different ways [7–9]. The solution can be expressed as a linear combination of the initial probabilities $p_n(0)$, supposed to be known, as

$$p_n(t) = \sum_{m=0}^{\infty} X_{n,m}(t)p_m(0). \tag{6}$$

The propagation matrix $X_{n,m}(t)$ for the probability distribution is found to be

$$X_{n,m}(t) = \frac{u^n}{(1+u)^{n+1}}\left(\frac{1+v}{1+u}\right)^m \sum_k \frac{(m+n-k)!}{(n-k)!k!(m-k)!}\left[-\frac{v(1+u)}{u(1+v)}\right]^k, \tag{7}$$

where we introduced the abbreviations

$$u = n_{eq}\{1-\mathrm{e}^{-Kt}\}, \quad v = n_{eq} - (n_{eq}+1)\mathrm{e}^{-Kt}. \tag{8}$$

The statistical properties of the photon field are more conveniently expressed in terms of the factorial moments $s_k(t)$, defined by

$$s_k(t) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} p_n(t), \tag{9}$$

for $k = 0, 1, 2, \ldots$ For $k = 0$ we have $s_0(t) = 1$, which is the normalization of the probability distribution $p_n(t)$. The factorial moment $s_1(t)$ equals the average number of photons in the cavity at time t, and we shall indicate this by $\bar{n}(t)$. Then, since $s_2(t)$ equals the average of $n(n-1)$, it is related to the variance of the photon number distribution as

$$s_2(t) = \mathrm{var}(t) + \bar{n}(t)(\bar{n}(t) - 1). \tag{10}$$

The factorial moments can be calculated, and the result is found to be [8]

$$s_k(t) = \sum_{\ell=0}^{k} \left(\frac{k!}{\ell!}\right)^2 \frac{1}{(k-\ell)!} \left[n_{\mathrm{eq}}(1 - \mathrm{e}^{-Kt})\right]^{k-\ell} \mathrm{e}^{-\ell K t} s_\ell(0), \tag{11}$$

expressed in the initial factorial moments $s_k(0)$. For $k = 1$ and $k = 2$ this yields

$$s_1(t) = n_{\mathrm{eq}} + (n_{\mathrm{o}} - n_{\mathrm{eq}})\mathrm{e}^{-Kt}, \tag{12}$$

$$s_2(t) = 2n_{\mathrm{eq}}^2 + 4n_{\mathrm{eq}}(n_{\mathrm{o}} - n_{\mathrm{eq}})\mathrm{e}^{-Kt} + \left\{s_2(0) + 2n_{\mathrm{eq}}(n_{\mathrm{eq}} - 2n_{\mathrm{o}})\right\}\mathrm{e}^{-2Kt}, \tag{13}$$

where we have set $n_{\mathrm{o}} = \bar{n}(0)$. Also of interest is the limit of thermal equilibrium, $t \to \infty$, for which we have

$$s_k(\infty) = k! n_{\mathrm{eq}}^k. \tag{14}$$

An appealing normalized representation of the width of the photon number distribution is the q-factor, defined as

$$q(t) = \frac{\mathrm{var}(t) - \bar{n}(t)}{\bar{n}(t)}. \tag{15}$$

In terms of the factorial moments, $q(t)$ can be expressed as

$$q(t) = \frac{s_2(t) - s_1(t)^2}{s_1(t)}, \tag{16}$$

and with Eqs. (12) and (13) an explicit expression can be obtained. From definition (15) we have $q(t) \geq -1$. For a Poisson distribution we have $\mathrm{var}(t) = \bar{n}(t)$, and therefore $q(t) = 0$. Any field for which $q(t) < 0$ is called sub-poisson, since the photon distribution is narrower than a Poisson distribution with the same average number of photons in the field. With Eq. (14) we find that in thermal equilibrium we have

$$q(\infty) = n_{\mathrm{eq}} > 0, \tag{17}$$

showing that the distribution becomes super-poisson in the long-time limit for non-zero temperature. Therefore, if the initial distribution is sub-poisson, $q(0) \equiv q_{\mathrm{o}} < 0$, there must be an instant t_{P} at which we have $q(t_{\mathrm{P}}) = 0$. This 'Poisson point' occurs at

$$t_{\mathrm{P}} = \frac{1}{K} \ln\left\{1 - \frac{n_{\mathrm{o}}}{n_{\mathrm{eq}}}\left(1 - \sqrt{1 - \frac{q_{\mathrm{o}}}{n_{\mathrm{o}}}}\right)\right\}, \tag{18}$$

as can be derived from the expressions above. In other words, t_{P} is the survival time of a sub-poisson distribution in a cavity.

For zero temperature, $n_{eq}=0$, the factorial moments simplify to

$$s_k(t)=s_k(0)e^{-kKt}, \tag{19}$$

and this yields for the q-factor

$$q(t)=q_o e^{-Kt}. \tag{20}$$

In this limit, a sub(super)-poisson distribution remains sub(super)-poisson at all times.

3. Intensity correlations

In order to evaluate the intensity correlations, we first transform expression (4) to the Schrödinger picture, which yields

$$I_k(t_1,\dots,t_k)=\zeta^k Tr\boldsymbol{D}e^{-iL(t_k-t_{k-1})}\boldsymbol{D}\dots\boldsymbol{D}e^{-iL(t_2-t_1)}\boldsymbol{D}\rho(t_1). \tag{21}$$

The Liouville operator $\boldsymbol{D}$ is defined by

$$\boldsymbol{D}\Pi=a\Pi a^{\dagger}, \tag{22}$$

showing its effect on an arbitrary Hilbert space operator Π.

The uncorrelated intensity $(k=1)$ is

$$I_1(t_1)=\zeta Tr\boldsymbol{D}\rho(t_1). \tag{23}$$

From the definition of $\boldsymbol{D}$ we obtain the relation

$$Tr\boldsymbol{D}\Pi=\sum_{n=0}^{\infty}n\langle n|\Pi|n\rangle, \tag{24}$$

for an arbitrary operator Π. Then for $\Pi=\rho(t_1)$ we have $\langle n|\rho(t_1)|n\rangle=p_n(t_1)$, and from this

$$I_1(t_1)=\zeta\bar{n}(t_1)=\zeta\left(n_{eq}+(n_o-n_{eq})e^{-Kt_1}\right). \tag{25}$$

Therefore, the photon detection rate I_1 at time t_1 is simply equal to the average number of photons in the cavity at time t_1, times the detection efficiency parameter ζ. Or we can say that ζdt_1 equals the probability for a photon in the cavity to be detected in $[t_1,t_1+dt_1]$, and this probability is independent of time and of the state of the radiation field.

For the evaluation of the higher order correlations we proceed in a similar way. We work from the outside in, and start with Eq. (24). Then we replace Π by $\exp(-iLt)\Pi$ with $t\equiv t_k-t_{k-1}$, and use the relation

$$\langle n|(e^{-iLt}\Pi)|n\rangle=\sum_{m=0}^{\infty}X_{n,m}(t)\langle m|\Pi|m\rangle, \tag{26}$$

which follows from the fact that $X_{n,m}(t)$ equals the conditional probability for having n photons at time t, if we have m photons at time zero, or from Eq. (6). Then we replace Π by $\boldsymbol{D}\Pi$ and use

$$\langle m|\boldsymbol{D}\Pi|m\rangle=(m+1)\langle m+1|\Pi|m+1\rangle, \tag{27}$$

and so on. This then yields the expression

$$I_k(t_1,..,t_k)=\zeta^k\sum_{n_k=1}^{\infty}..\sum_{n_2=1}^{\infty}\sum_{n_1=1}^{\infty}n_k..n_2n_1X_{n_k,n_{k-1}-1}(t_k-t_{k-1})..X_{n_2,n_1-1}(t_2-t_1)p_{n_1}(t_1). \tag{28}$$

Finally we need to perform this k-fold summation. To this end, we first notice that $I_1(t_1)$ can be written as

$$I_1(t_1) = \zeta s_1(t_1). \tag{29}$$

For $k=2$ we have

$$I_2(t_1,t_2) = \zeta^2 \sum_{n_2=1}^{\infty} \sum_{n_1=1}^{\infty} n_2 n_1 X_{n_2,n_1-1}(t_2-t_1) p_{n_1}(t_1). \tag{30}$$

We sum over n_2 first. Since $X_{n_2,n_1-1}(t_2-t_1)$ equals $p_{n_2}(t_2-t_1)$ under the condition that there are n_1-1 photons at time zero, the summation over n_2 equals $\bar{n}(t_2-t_1)$ under condition $n_o = n_1 - 1$. With Eq. (12) we therefore have

$$\sum_{n_2=1}^{\infty} n_2 X_{n_2,n_1-1}(t_2-t_1) = n_{eq}(1-e^{-K(t_2-t_1)}) + (n_1-1)e^{-K(t_2-t_1)}. \tag{31}$$

Then times $n_1 p_{n_1}(t_1)$ and sum over n_1. The first term in Eq. (31) then gives $\bar{n}(t_1)$, which is $s_1(t_1)$, and the factor $n_1(n_1-1)$ in the second term gives $s_2(t_1)$. The result for $I_2(t_1,t_2)$ is therefore

$$I_2(t_1,t_2) = \zeta^2 n_{eq} s_1(t_1)(1-e^{-K(t_2-t_1)}) + \zeta^2 s_2(t_1) e^{-K(t_2-t_1)}. \tag{32}$$

After inserting Eqs. (12) and (13) for the factorial moments, we can then obtain the full time dependence of the two-photon correlation function.

Working out the $k=3$ correlation function in the same way as above yields the result

$$I_3(t_1,t_2,t_3) = \zeta^3 n_{eq}^2 s_1(t_1)(1-e^{-K(t_2-t_1)})(1+e^{-K(t_3-t_2)}-2e^{-K(t_3-t_1)}) + \zeta^3 s_2(t_1) n_{eq}(1+3e^{-K(t_3-t_2)} -4e^{-K(t_3-t_1)})e^{-K(t_2-t_1)} + \zeta^3 s_3(t_1) e^{-K(t_3-t_1)} e^{-K(t_2-t_1)}. \tag{33}$$

It now becomes clear what the general structure of the correlation functions is. For arbitrary k, they can be written as

$$I_k(t_1,\ldots,t_k) = \zeta^k \sum_{m=1}^{k} n_{eq}^{k-m} s_m(t_1) \tilde{Z}_{k,m}(t_1,\ldots,t_k). \tag{34}$$

The functions $\tilde{Z}_{k,m}(t_1,\ldots,t_k)$ are independent of the temperature parameter n_{eq}, and independent of the state of the system at time t_1 or time zero. They are combinatorial functions of the k time variables only. Hence

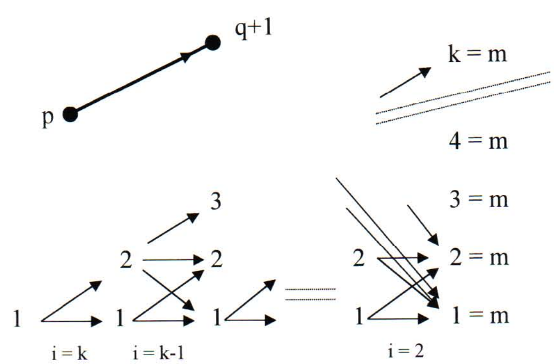

Fig. 1. The combinatorics lattice for the evaluation of $\tilde{Z}_{k,m}$ in Eq. (35). The size of the lattice is determined by k. For the computation of $\tilde{Z}_{k,m}$ one starts at the lower left corner and goes in unit steps to the desired m value on the right. The only restriction is that in each step one can go up by only one level. Each step determines a factor in the product in Eq. (35), and the values of p and q are determined by the numbers that are connected by the arrow. For convenience we indicate the final number by $q+1$, rather than q. The values of i, written below the lattice, determine the time arguments t_i and t_{i-1} in Eq. (35). The summation in Eq. (35) then runs over all possible paths on this lattice.

expression (34) displays the temperature dependence and the dependence on the state of the system, which appears to enter through the factorial moments only.

For the combinatorial functions we have found the following result:

$$\tilde{Z}_{k,m}(t_1,\ldots,t_k) = \sum_{\substack{\text{all paths} \\ 1\to m}} \prod_{i=k}^{2} \left(\frac{p!}{q!}\right)^2 \frac{1}{(p-q)!}\left(1-\mathrm{e}^{-K(t_i-t_{i-1})}\right)^{p-q}\mathrm{e}^{-qK(t_i-t_{i-1})}. \tag{35}$$

The summation runs over all paths on the lattice shown in Fig. 1. We start at '1' in the lower left corner, and go in unit steps to the desired m value on the right. In each step you can go up one level, remain horizontal, or go down by an arbitrary number of levels. Each step determines a factor in the product. The values of p and q follow from the numbers on the lattice that are connected, according to the inserted diagram. Combination of Eqs. (34) and (35) then determines the kth order photon correlation, and this is the main result of this paper.

4. Representation in terms of the initial state

The factorial moments $s_m(t_1)$ in Eq. (34) can be expressed in terms of the initial factorial moments $s_n(0)$ by means of Eq. (11). When we then rearrange the powers of n_{eq} to obtain an expression similar to Eq. (34), we find

$$I_k(t_1,\ldots,t_k) = \zeta^k \sum_{n=0}^{k} n_{\mathrm{eq}}^{k-n} s_n(0) Z_{k,n}(t_1,\ldots,t_k), \tag{36}$$

in terms of different combinatorial functions $Z_{k,n}$. The relation between the two sets of functions is

$$Z_{k,n}(t_1,\ldots,t_k) = \sum_{m=n}^{k} \left(\frac{m!}{n!}\right)^2 \frac{1}{(m-n)!}\left(1-\mathrm{e}^{-Kt_1}\right)^{m-n}\mathrm{e}^{-nKt_1}\tilde{Z}_{k,m}(t_1,\ldots,t_k), \tag{37}$$

where we have set $\tilde{Z}_{k,0}=0$. The combinatorial functions $Z_{k,n}$ are considerably more complicated the functions $\tilde{Z}_{k,m}$, as illustrated below.

Without calculating the combinatorial functions explicitly, a number of interesting relations can be derived. Let x be an auxiliary parameter to make a generating function for the $Z_{k,n}$'s. With Eq. (37) we then derive

$$\sum_{n=0}^{k} n! Z_{k,n} x^n = \sum_{m=0}^{k} m! \tilde{Z}_{k,m}\left(1+(x-1)\mathrm{e}^{-Kt_1}\right)^m. \tag{38}$$

For instance, setting $x=1$ yields the symmetric relation

$$\sum_{n=0}^{k} n! Z_{k,n} = \sum_{m=0}^{k} m! \tilde{Z}_{k,m}, \tag{39}$$

and another one is

$$\sum_{n=0}^{k} n! Z_{k,n}\left(-\mathrm{e}^{Kt_1}\right)^n = \sum_{m=0}^{k} m! \tilde{Z}_{k,m}\left(-\mathrm{e}^{-Kt_1}\right)^m. \tag{40}$$

By letting $x = 1-\exp(Kt_1)$ we find a sum rule for $Z_{k,n}$:

$$\sum_{n=0}^{k} n!\left(1-\mathrm{e}^{Kt_1}\right)^n Z_{k,n} = 0. \tag{41}$$

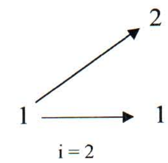

Fig. 2. The $k=2$ lattice and possible paths. The path $1 \to 1$ gives the only contribution to $\tilde{Z}_{2,1}$, and for this path we have $p=1$ and $q=0$. Similarly, the path $1 \to 2$ determines $\tilde{Z}_{2,2}$ (with $p=q=1$).

Result (38) can also be used to find the $\tilde{Z}_{k,m}$'s in terms of the $Z_{k,n}$'s, which is the inverse relation of Eq. (37). To this end, set $y = 1 + (x-1)\exp(-Kt_1)$ in (38), which gives

$$\sum_{\ell=0}^{k} \ell!\tilde{Z}_{k,\ell} y^{\ell} = \sum_{n=0}^{k} n! Z_{k,n} \left(1 + (y-1)e^{Kt_1}\right)^n. \tag{42}$$

Then the $\tilde{Z}_{k,m}$'s can be found through

$$\tilde{Z}_{k,m} = \frac{1}{(m!)^2} \left\{ \left(\frac{\partial}{\partial y} \right)^m \sum_{\ell=0}^{k} \ell!\tilde{Z}_{k,\ell} y^{\ell} \right\}_{y=0}, \tag{43}$$

with result

$$\tilde{Z}_{k,m}(t_1,\ldots,t_k) = \sum_{n=m}^{k} \left(\frac{n!}{m!} \right)^2 \frac{1}{(n-m)!} \left(1 - e^{Kt_1}\right)^{n-m} e^{mKt_1} Z_{k,n}(t_1,\ldots,t_k). \tag{44}$$

It is interesting to note that if we set $m=0$ here, we recover again Eq. (41), since $\tilde{Z}_{k,0} = 0$.

5. Combinatorial functions

For $k=1$ the lattice reduces to one point, and we have $\tilde{Z}_{1,1} = 1$. With Eq. (34) this gives $I_1(t_1) = \zeta s_1(t_1)$, as in Eq. (29). With Eq. (37) we find the other combinatorial functions to be

$$Z_{1,0} = 1 - e^{-Kt_1},\ Z_{1,1} = e^{-Kt_1}, \tag{45}$$

and with Eq. (36) we then get

$$I_1(t_1) = \zeta s_1(t_1) = \zeta\left(n_{eq}(1 - e^{-Kt_1}) + n_o e^{-Kt_1}\right), \tag{46}$$

as before. Here we have set $s_0(0) = 1$ and $s_1(0) = n_o$.

For $k=2$ the lattice is shown in Fig. 2. Each product in Eq. (35) has one factor and each summation has one term. This yields

$$\tilde{Z}_{2,1} = 1 - e^{-K(t_2-t_1)}, \quad \tilde{Z}_{2,2} = e^{-K(t_2-t_1)}, \tag{47}$$

and with Eq. (34) this leads to expression (32) for $I_2(t_1,t_2)$. The combinatorial functions that refer to the initial state of the field are then

$$Z_{2,0} = (1 - e^{-K(t_2-t_1)})(1 - e^{-Kt_1}) + 2e^{-K(t_2-t_1)}(1 - e^{-Kt_1})^2, \tag{48}$$

$$Z_{2,1} = (1 - e^{-K(t_2-t_1)})e^{-Kt_1} + 4e^{-K(t_2-t_1)}(1 - e^{-Kt_1})e^{-Kt_1}, \tag{49}$$

$$Z_{2,2} = e^{-K(t_2-t_1)}e^{-2Kt_1}. \tag{50}$$

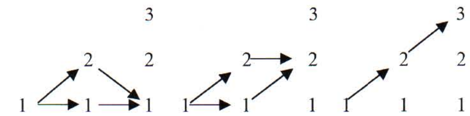

Fig. 3. Diagrams (a), (b) and (c) show the contributing paths on the $k=3$ lattice for the determination of $\tilde{Z}_{3,1}$, $\tilde{Z}_{3,2}$ and $\tilde{Z}_{3,3}$, respectively.

The corresponding expression for $I_2(t_1,t_2)$ becomes, after simplifying,

$$I_2(t_1,t_2) = \zeta^2 n_{eq}\left(n_{eq} + (n_0 - n_{eq})e^{-Kt_1}\right) + \zeta^2 e^{-K(t_2-t_1)}\Big(n_{eq}^2 + 3n_{eq}(n_o - n_{eq})e^{-Kt_1} + \left(s_2(0) + 2n_{eq}(n_{eq} - 2n_o)\right)e^{-2Kt_1}\Big). \tag{51}$$

This result has been derived before in many different ways [10,11].

For $k>2$ the expressions become more involved very rapidly. Fig. 3 shows the lattice paths for the three-photon correlation function, and we obtain, after some reorganizing,

$$\tilde{Z}_{3,1} = (1 - e^{-K(t_2-t_1)})(1 + e^{-K(t_3-t_2)} - 2e^{-K(t_3-t_1)}), \tag{52}$$

$$\tilde{Z}_{3,2} = (1 + 3e^{-K(t_3-t_2)} - 4e^{-K(t_3-t_1)})e^{-K(t_2-t_1)}, \tag{53}$$

$$\tilde{Z}_{3,3} = e^{-K(t_3-t_2)}e^{-2K(t_2-t_1)}, \tag{54}$$

from which the $Z_{3,n}$'s can be found.

It seems that the only higher order combinatorial function which has a simple form is $Z_{k,k}$. The only contributing path on the lattice runs from the lower-left '1' to $m=k$ on the top of the lattice. We then find

$$\tilde{Z}_{k,k} = e^{-K(t_k + t_{k-1} + \ldots + t_2 - (k-1)t_1)}, \tag{55}$$

and from this

$$Z_{k,k} = e^{-K(t_k + t_{k-1} + \ldots + t_2 + t_1)}, \tag{56}$$

with Eq. (37).

6. Special cases

An interesting limit is the case of zero temperature, for which $n_{eq}=0$. We then have $n_{eq}^{k-n} = \delta_{n,k}$, so that in Eq. (36) only the term with $n=k$ survives. With Eq. (56) this yields the simple result

$$I_k(t_1,\ldots,t_k) = \zeta^k s_k(0) e^{-Kt_k} \ldots e^{-Kt_2} e^{-Kt_1}. \tag{57}$$

This factorization allows the interpretation that $\exp(-Kt)$ is the probability that a photon which is present in the cavity at $t=0$ will still be present at time t, and can then be detected, with ζ being the detection rate for each photon.

Especially simple is the case of a free field, for which there is no damping. If we set $K=0$ in Eq. (35) for the combinatorial functions, we see that each combinatorial factor can only be non-zero for $p=q$. The only contribution to the summation comes from the path $1 \to 2 \to \ldots \to m=k$, along the upper diagonal, and therefore we find

$$\tilde{Z}_{k,m}(t_1,\ldots,t_k) = \delta_{m,k}. \tag{58}$$

With Eq. (34) we then find for the intensity correlations

$$I_k(t_1,\ldots,t_k) = \zeta^k s_k(t_1). \tag{59}$$

On the other hand, with Eq. (37) with $K=0$ and Eq. (58), we have

$$Z_{k,n}(t_1,\ldots,t_k)=\delta_{n,k}, \tag{60}$$

which then gives with Eq. (36)

$$I_k(t_1,\ldots,t_k)=\zeta^k s_k(0). \tag{61}$$

Both expressions for $I_k(t_1,\ldots,t_k)$ are consistent with the fact that for a free field the photon probability distribution is independent of time. Or, for a free field we have $a(t)=a\exp(-\mathrm{i}\,\omega_c t)$ and $a^\dagger(t)=a^\dagger\exp(\mathrm{i}\,\omega_c t)$, and therefore all exponentials in expression (4) cancel.

As another example of a special limit we consider the equal time correlations $I_k(t,\ldots,t)$. When we set $t_1=\ldots=t_k\equiv t$ in Eq. (21) we obtain

$$I_k(t,\ldots,t)=\zeta^k Tr\boldsymbol{D}^k\rho(t), \tag{62}$$

and with definition (22) of $\boldsymbol{D}$ this becomes

$$I_k(t,\ldots,t)=\zeta^k Tr\rho(t)(a^\dagger)^k a^k, \tag{63}$$

and this is

$$I_k(t,\ldots,t)=\zeta^k s_k(t). \tag{64}$$

On the other hand, if we set all times equal in the combinatorial Eq. (35), we get again only contributions from steps for which $p=q$. Therefore we have $\tilde{Z}_{k,m}(t,\ldots,t)=\delta_{m,k}$, and with Eq. (34) this also yields the result (64). The equal-time functions $Z_{k,n}(t,\ldots,t)$ become with Eq. (37):

$$Z_{k,n}(t,\ldots,t)=\left(\frac{k!}{n!}\right)^2\frac{1}{(k-n)!}\left(1-\mathrm{e}^{-Kt}\right)^{k-n}\mathrm{e}^{-nKt}. \tag{65}$$

When substituted in Eq. (36), this gives again (64), since the resulting summation is just expression (11) for the factorial moments.

Finally, we consider the limit of thermal equilibrium. We let $t_1\to\infty$, but keep all time differences finite. The factorial moments are then given by Eq. (14), and the intensity correlations become with Eq. (34)

$$I_k(t_1,\ldots,t_k)=(\zeta n_{\mathrm{eq}})^k\sum_{m=1}^{k} m!\tilde{Z}_{k,m}(t_1,\ldots,t_k). \tag{66}$$

Here the temperature dependence truly factors out as an overall constant $n_{\mathrm{eq}}^{\mathrm{k}}$, whereas in the general result Eq. (34) it only factors out for each term in the summation. It should be noted that we could equally have replaced the $s_n(0)$ in Eq. (36) by its equilibrium value, which leads again to Eq. (66), after using identity (39).

7. Photon detection statistics

Let $P_n(t)$ be the probability for the detection of n photons from the radiation in the time interval $[0,t]$, and let $S_k(t)$ be the corresponding factorial moments, defined as in Eq. (9) for the field statistics. The factorial moments are determined by the field intensity correlations according to [12]

$$S_k(t)=k!\int_0^t \mathrm{d}t_k\int_0^{t_k}\mathrm{d}t_{k-1}\ldots\int_0^{t_2}\mathrm{d}t_1\, I_k(t_1,\ldots,t_k). \tag{67}$$

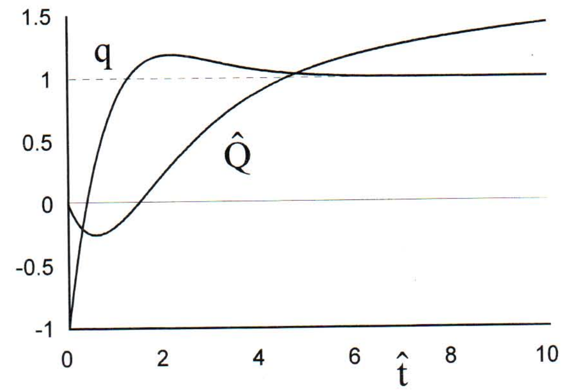

Fig. 4. Illustration of the time dependence of q and $\hat{Q}$. The parameters are $n_o = 5$, $n_{eq} = 1$ and $q_o = -1$. The Poisson point, given by Eq. (18), for these values is $\tilde{t}_P = 0.39$. We see that q reaches its steady-state value of $q(\infty) = 1$ very rapidly, but $\hat{Q}$ only tends to $\hat{Q}(\infty) = 2$ on a long time scale. The slope of the graph of $\hat{Q}$ at $\hat{t} = 0$ equals q_o.

The average number of detected photons in $[0,t]$, indicated by $\mu(t)$, equals $S_1(t)$, and with expression (25) for $I_1(t_1)$ we readily obtain

$$S_1(t) = \mu(t) = \frac{\zeta}{K}\left\{n_{eq}\hat{t} + (n_o - n_{eq})(1 - e^{-\hat{t}})\right\}, \tag{68}$$

where we have introduced the abbreviation $\hat{t} = Kt$. The variance of the photon count distribution, denoted by $\sigma^2(t)$, is determined by the second factorial moment according to

$$\sigma^2(t) = S_2(t) - S_1(t)[S_1(t) - 1], \tag{69}$$

just as in Eq. (10) for the field statistics. With the result (32) for $I_2(t_1,t_2)$ we then find for $S_2(t)$:

$$S_2(t) = \frac{\zeta^2}{K^2}\left\{n_{eq}^2\hat{t}^2 + 2n_{eq}n_o\hat{t} + 2n_{eq}(2n_o - 3n_{eq})(1 - e^{-\hat{t}}) - 6n_{eq}(n_o - n_{eq})\hat{t}e^{-\hat{t}} \right.$$
$$\left. + \left[s_2(0) + 2n_{eq}(n_{eq} - 2n_o)\right](1 - e^{-\hat{t}})^2\right\}. \tag{70}$$

As a normalized measure for the variance we also introduce here the Q-factor:

$$Q(t) = \frac{\sigma^2(t) - \mu(t)}{\mu(t)}, \tag{71}$$

and in terms of the factorial moments this becomes

$$Q(t) = \frac{S_2(t) - S_1(t)^2}{S_1(t)}, \tag{72}$$

similar to expression (16). With Eqs. (68) and (70) we then obtain $Q(t)$. It follows from Eq. (72), and Eqs. (68) and (70), that $Q(t)$ only depends on the detection parameter ζ through an overall factor of ζ/K. We therefore define $\hat{Q}(t) = Q(t)/(\zeta/K)$, which is independent of the photon detection rate, and determined by field properties only.

It is interesting to notice that for small $\hat{t}$ both $\mu(t)$ and $\sigma^2(t)$ are $\mathscr{O}(\hat{t})$ but the difference is $\mathscr{O}(\hat{t}^2)$, and therefore $\hat{Q}(t) = \mathscr{O}(\hat{t})$. To be precise, the small-time behavior of $\hat{Q}(t)$ is found to be

$$\hat{Q}(t) = q_o\hat{t} + \ldots, \tag{73}$$

with q_o the q-factor of the field statistics at t = 0. On the other hand, for $\hat{t} \to \infty$ we have for the detection statistics $\hat{Q}(\infty) = 2n_{eq}$, whereas for the field statistics we have $q(\infty) = n_{eq}$. Fig. 4 shows the typical behavior of q and $\hat{Q}$ as a function of $\hat{t}$. It is seen that the field statistics becomes superpoisson well before the counting statistics. Also, the field q factor reaches its steady-state value exponentially, whereas the detection $\hat{Q}$ factor approaches its long-time limit extremely slowly.

8. Conclusions

We have evaluated the photon intensity correlation functions for radiation in a damped single-mode cavity. It appeared that these correlation functions can be expressed in the form given by Eq. (34). The temperature dependence arises as an overall factor in each term of the summation, and the dependence on the initial state enters through the factorial moments of the photon number distribution at time t_1. The temporal correlations are determined entirely by the combinatorial functions $\tilde{Z}_{k,m}(t_1,\ldots,t_k)$, and these were evaluated explicitly, with the result given by Eq. (35). The form of $\tilde{Z}_{k,m}(t_1,\ldots,t_k)$ is a combinatorial sum over paths on the lattice shown in Fig. 1. Each step on the lattice determines the numbers p, q and i, which then in turn determine the combinatorial factor given in Eq. (35). The combinatorial functions depend only on the time arguments, and not on the temperature, nor on the density operator of the radiation field. An equivalent representation is given by Eq. (36), expressing the photon correlations in terms of the factorial moments at time zero. This result involves a different set of combinatorial functions, which can be found from $\tilde{Z}_{k,m}(t_1,\ldots,t_k)$ with the help of Eq. (37). It should be noted that the $\tilde{Z}_{k,m}(t_1,\ldots,t_k)$'s only depend on the time arguments through the time differences between consecutive detection times, whereas the $Z_{k,n}(t_1,\ldots,t_k)$'s also depend explicitly on t_1, the time at which the first photon is detected. We have derived various relations between the two sets of combinatorial functions in Section 4. In Section 6 we have shown that our result leads to known expressions in limiting cases, and in Section 7 we have illustrated how the photon correlations can be used to study the photon detection statistics.

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