Available online at www.sciencedirect.com
SCIENCE DIRECT®
Surface Science 590 (2005) 101–116
SURFACE SCIENCE
www.elsevier.com/locate/susc

Angular spectrum representation of the electromagnetic multipole fields, and their reflection at a perfect conductor

Henk F. Arnoldus *

Department of Physics and Astronomy, Mississippi State University, P.O. Drawer 5167, Mississippi State, MS 39762-5167, USA

Received 11 January 2005; accepted for publication 8 June 2005
Available online 6 July 2005

Abstract

Angular spectrum representations are derived for electric and magnetic multipole fields of arbitrary order. The result involves generalized spherical harmonics and generalized vector spherical harmonics, and the representations are in the form of integrals over the $\mathbf{k}_{\parallel}$-plane. The representations are especially useful for the study of reflection and transmission of multipole radiation by a plane interface. As an example, we have considered the reflection at a perfect conductor. The reflected field of a multipole field could be expressed in the form of an angular spectrum with a very simple relation to the angular spectrum of the source field. The radiation pattern of a multipole near the perfect conductor is obtained with the method of stationary phase. We also introduce a method for determining the mirror image of the source of an arbitrary multipole.

Keywords: Light scattering; Photon emission; Metallic surfaces; Metal–insulator interfaces

1. Introduction

Electromagnetic radiation emitted by a localized source can most conveniently be represented as a superposition of electromagnetic multipole fields [1]. In particular, when the source is an atom [2] or a nucleus [3] then emission of radiation occurs during a transition between states with well-defined angular momentum and the emitted radiation is a pure multipole field of order (ℓ, m), where ℓ and m are determined by the angular momentum quantum numbers of the initial and final states. Multipole radiation comes in two types, electric and magnetic, with electric dipole $(\ell = 1)$ and electric quadrupole $(\ell = 2)$ radiation encountered most commonly.

When the source is located in the vicinity of a material medium like a dielectric or a metal substrate, then the emitted radiation will be reflected

* Tel.: +1 662 325 2919; fax: +1 662 325 8898.
E-mail address: arnoldus@ra.msstate.edu

0039-6028/$ - see front matter
doi:10.1016/j.susc.2005.06.012

by and transmitted through the interface. This problem of reflection and transmission has been studied extensively for dipole radiation [4–6]. Of particular interest is the situation where the distance between the source and the interface is of the order of a wavelength of the radiation, since this leads to interesting interference patterns. Furthermore, when the source is an atom (or molecule), the reflected radiation interacts with the atom, and this affects the time evolution of the atomic density operator. As a result, the lifetime of the excited state is altered, and this leads to enhancement or inhibition of the spontaneous emission [7–15].

Multipole fields are spherical waves emanating from the site of the source. Such a representation does not allow in any simple way to study the reflection and transmission by an interface. A better approach to this problem is to adopt an angular spectrum representation of the radiation, which is a superposition of plane waves, each of which obeys Maxwell's equations individually. The reflection and transmission of each partial wave of the spectrum is then accounted for by an appropriate Fresnel coefficient, and the reflected and transmitted fields are again represented by angular spectra [16]. This method has been applied successfully for radiation emitted by electric and magnetic dipoles for which the angular spectrum representation is known [17,18]. In this paper we extend this approach to multipoles of arbitrary order (ℓ, m) and type (electric and magnetic).

For radiation with angular frequency ω, propagating in a medium with index of refraction n, a plane wave has the form $\exp(\mathrm{i}\boldsymbol{K}_\pm \cdot \boldsymbol{r})$, with the wave vector given by

$$\boldsymbol{K}_\pm = \boldsymbol{k}_\parallel \pm \beta \boldsymbol{e}_z. \tag{1}$$

In an angular spectrum of plane waves, the z-axis is a preferred direction in space, and the xy-plane is seen as separating the regions $z > 0$ and $z < 0$. In Eq. (1), the upper (lower) sign holds for waves in the region $z > 0$ $(z < 0)$, and the subscript $\parallel$ refers to the orientation with respect to the xy-plane. The vector $\boldsymbol{k}_\parallel$ is a free parameter in the angular spectrum (the integration variable), and is allowed to have any real value. The dispersion relation for the waves in the medium is $\boldsymbol{K}_\pm \cdot \boldsymbol{K}_\pm = n^2 k_0^2$, with

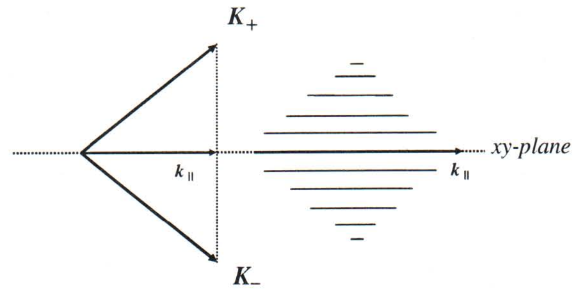

Fig. 1. Schematic illustration of the plane-wave modes of the angular spectrum. For a given $\boldsymbol{k}_\parallel$, with magnitude $k_\parallel$, the wave is traveling for $k_\parallel < nk_0$, and this is indicated by the vectors $\boldsymbol{K}_+$ and $\boldsymbol{K}_-$. The wave travels in the directions away from the xy-plane. For $k_\parallel > nk_0$, the z-component of $\boldsymbol{K}_\pm$ is imaginary and the wave decays away from the xy-plane while traveling along the xy-plane, as shown on the right.

$k_0 = \omega/c$, and therefore parameter β in Eq. (1) is given by

$$\beta = \begin{cases} \sqrt{n^2k_0^2 - k_\parallel^2}, & k_\parallel < nk_0, \\ \mathrm{i}\sqrt{k_\parallel^2 - n^2k_0^2}, & k_\parallel > nk_0. \end{cases} \tag{2}$$

For $k_\parallel < nk_0$, β is real and positive, and a wave with wave vector $\boldsymbol{K}_\pm$ is a traveling wave, which travels in a direction away from the xy-plane. On the other hand, for $k_\parallel > nk_0$, parameter β is positive imaginary, and a plane wave of the form $\exp(\mathrm{i}\boldsymbol{K}_\pm \cdot \boldsymbol{r})$ travels along the xy-plane in the direction of $\boldsymbol{k}_\parallel$, but decays in the z-direction away from the xy-plane at both sides. These are the evanescent waves of the angular spectrum. Fig. 1 shows schematically the traveling and evanescent waves in the angular spectrum.

2. Scalar multipole fields

Before considering the electromagnetic multipole fields, we first consider the scalar multipole fields $\Pi_{\ell m}(\boldsymbol{r})$. These fields are solutions of the Helmholz equation

$$(\nabla^2 + n^2k_0^2)\Pi_{\ell m}(\boldsymbol{r}) = 0, \tag{3}$$

with outgoing boundary conditions at infinity. In spherical coordinates (r, θ, ϕ) the solutions are

$$\Pi_{\ell m}(\mathbf{r}) = h_\ell^{(1)}(nk_0r)Y_{\ell m}(\theta,\phi), \tag{4}$$

in terms of a spherical Hankel function $h_\ell^{(1)}(nk_0r)$ and a spherical harmonic $Y_{\ell m}(\theta,\phi)$. For a given $\ell = 0,1,2,\ldots$, the values of m are $m = -\ell,\ldots,\ell$, and the lowest-order scalar multipole is

$$\Pi_{00}(\mathbf{r}) = \frac{1}{\sqrt{4\pi}}\frac{e^{ink_0r}}{ink_0r}. \tag{5}$$

The higher order multipoles can be obtained from $\Pi_{00}(\mathbf{r})$ through differentiation by means of a theorem due to Erdélyi [19]

$$\Pi_{\ell m}(\mathbf{r}) = (-i)^\ell(-1)^m\sqrt{\frac{(2\ell+1)(\ell-m)!}{(\ell+m)!}} \times\left\{\left[\frac{1}{ink_0}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]^m R_{\ell m}\left(\frac{1}{ink_0}\frac{\partial}{\partial z}\right)\right\}\Pi_{00}(\mathbf{r}), \quad m \geqslant 0. \tag{6}$$

Here,

$$R_{\ell m}(u) = \frac{d^m}{du^m}P_\ell(u) \tag{7}$$

and $P_\ell(u)$ is a Legendre polynomial.

An angular spectrum representation of $\Pi_{\ell m}(\mathbf{r})$ can be obtained [20] by noticing that $\Pi_{00}(\mathbf{r})$ is the scalar Green's function, apart from a constant, for which the angular spectrum is given by Weyl's representation [21]. We have

$$\Pi_{00}(\mathbf{r}) = \frac{1}{4\pi^{3/2}nk_0}\int d^2\mathbf{k}_\parallel \frac{1}{\beta}e^{i\mathbf{K}_\pm\cdot\mathbf{r}} \tag{8}$$

in the notation of the previous section. The integral over $\mathbf{k}_\parallel$ runs over the entire $\mathbf{k}_\parallel$-plane. The derivatives with respect to x, y and z in Eq. (6) only affect the factor $\exp(i\mathbf{K}_\pm\cdot\mathbf{r})$ in the integrand, and the differentiations are easily carried out. To this end, we adopt polar coordinates $(k_\parallel,\bar{\phi})$ in the $\mathbf{k}_\parallel$-plane:

$$\mathbf{k}_\parallel = k_\parallel(\mathbf{e}_x\cos\bar{\phi} + \mathbf{e}_y\sin\bar{\phi}) \tag{9}$$

and we introduce and "angle" $\bar{\theta}$ by

$$k_\parallel = nk_0\sin\bar{\theta}. \tag{10}$$

For a traveling wave we have $0 \leqslant \sin\bar{\theta} < 1$, and $\bar{\theta}$ is the polar angle of the wave vector $\mathbf{K}_+$. For an evanescent wave we have $\sin\bar{\theta} > 1$, but we shall still indicate $k_\parallel/nk_0$ by $\sin\bar{\theta}$, keeping in mind that no angle $\bar{\theta}$ is associated with $\sin\bar{\theta}$. Similarly, we write

$$\beta = nk_0\cos\bar{\theta}, \tag{11}$$

which, according to Eq. (2), is positive or positive imaginary. With this notation we have

$$\mathbf{K}_\pm = nk_0\sin\bar{\theta}(\mathbf{e}_x\cos\bar{\phi} + \mathbf{e}_y\sin\bar{\phi}) + nk_0\mathbf{e}_z\cos\bar{\theta}. \tag{12}$$

From Eq. (6) we then find

$$\Pi_{\ell m}(\mathbf{r}) = (-i)^\ell(-1)^m\sqrt{\frac{(2\ell+1)(\ell-m)!}{(\ell+m)!}}\frac{1}{4\pi^{3/2}nk_0} \times\int d^2\mathbf{k}_\parallel\frac{1}{\beta}e^{i\mathbf{K}_\pm\cdot\mathbf{r}}\left(-\sin\bar{\theta}e^{i\bar{\phi}}\right)^m R_{\ell m}(\pm\cos\bar{\theta}), \quad m \geqslant 0, \tag{13}$$

where again the upper (lower) sign holds for the region $z > 0$ ($z < 0$).

3. Generalized spherical harmonics

In order to arrive at a more compact representation we define associated Legendre functions of a complex variable by

$$P_\ell^m(z) = (-1)^m(1-z^2)^{m/2}R_{\ell m}(z), \quad m \geqslant 0 \tag{14}$$

and the definition is extended to all m by means of

$$P_\ell^{-m}(z) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(z). \tag{15}$$

In Eq. (14), the branch cut for the square root is taken along the negative real axis, so that the branch lines for $P_\ell^m(z)$ are $(-\infty,-1)$ and $(1,\infty)$. It should be noted that the $P_\ell^m(z)$ defined here are not the same as the standard associated Legendre functions $\mathrm{P}_\ell^m(z)$, to which they are related as $P_\ell^m(z) = (\pm i)^m\mathrm{P}_\ell^m(z)$, with $\pm$ the sign of the imaginary part of z. The difference arises from the difference in branch cut [22]. For later reference we note that $P_\ell^m(z)$ satisfies

$$P_\ell^m(-z) = (-1)^{\ell+m}P_\ell^m(z), \tag{16}$$

since $R_{\ell m}(-z) = (-1)^{\ell+m}R_{\ell m}(z)$. Recursion relations for $\mathrm{P}_\ell^m(z)$ carry over to relations for $P_\ell^m(z)$.

We shall need the following recursion between functions with different m value:

$$P_\ell^{m+2}(z) + 2(m+1)\frac{z}{\sqrt{1-z^2}}P_\ell^{m+1}(z) + (\ell - m)(\ell + m + 1)P_\ell^m(z) = 0, \tag{17}$$

where it is understood that $P_\ell^m(z) \equiv 0$ for $m > \ell$.

We now define the functions

$$S_{\ell m}(z, \alpha) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(z)\mathrm{e}^{\mathrm{i}m\alpha} \tag{18}$$

for z complex and α real. These are the generalized spherical harmonics because for $z = \cos\theta$ and $\alpha = \phi$ they reduce to the regular spherical harmonics

$$S_{\ell m}(\cos\theta, \phi) = Y_{\ell m}(\theta, \phi). \tag{19}$$

From Eq. (16) we find

$$S_{\ell m}(-z, \alpha) = (-1)^{\ell+m}S_{\ell m}(z, \alpha) \tag{20}$$

and Eq. (17) becomes

$$\sqrt{(\ell+m+1)(\ell-m)}\mathrm{e}^{-\mathrm{i}\alpha}S_{\ell,m+1}(z,\alpha) + \frac{2mz}{\sqrt{1-z^2}}S_{\ell,m}(z,\alpha) + \sqrt{(\ell-m+1)(\ell+m)}\mathrm{e}^{\mathrm{i}\alpha}S_{\ell,m-1}(z,\alpha) = 0. \tag{21}$$

Here we set $S_{\ell m}(z,\alpha) \equiv 0$ for $|m| > \ell$. The case $m = -\ell$ in Eq. (21) does not follow from Eq. (17) and has to be verified independently.

4. Scalar multipole fields, continued

When we set $z = \pm\cos\bar{\theta}$ in Eqs. (14) and (18), and $\alpha = \bar{\phi}$ in Eq. (18) we get just the $\bar{\theta}$ and $\bar{\phi}$ dependence of the integrand on the right-hand side of Eq. (13), apart from the exponential. Therefore, the angular spectrum attains the simple form

$$\Pi_{\ell m}(\mathbf{r}) = \frac{(-\mathrm{i})^\ell}{2\pi n k_0}\int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot\mathbf{r}}S_{\ell m}(\pm\cos\bar{\theta}, \bar{\phi}) \tag{22}$$

for $m \geqslant 0$. We shall now show that the same expression holds for all m.

When we replace m by $-m$ and α by $-\alpha$ in the definition (18) and use relation (15), we find immediately

$$S_{\ell m}(z, -\alpha) = (-1)^m S_{\ell,-m}(z, \alpha) \tag{23}$$

and therefore with Eq. (19)

$$Y_{\ell m}(\theta, -\phi) = (-1)^m Y_{\ell,-m}(\theta, \phi). \tag{24}$$

Then the multipole fields from Eq. (4) obey

$$\Pi_{\ell m}(r, \theta, -\phi) = (-1)^m \Pi_{\ell,-m}(r, \theta, \phi). \tag{25}$$

The ϕ dependence in the representation (22) only enters as $\mathbf{K}_\pm\cdot\mathbf{r} = nk_0 r[\sin\bar{\theta}\sin\theta(\cos\bar{\phi}\cos\phi + \sin\bar{\phi}\sin\phi) + \cos\bar{\theta}\cos\theta]$. We replace ϕ by $-\phi$, change the integration variable $\bar{\phi}$ to $\bar{\phi}' = 2\pi - \bar{\phi}$, and drop the prime on $\bar{\phi}'$. This yields for $m \geqslant 0$

$$\Pi_{\ell m}(r, \theta, -\phi) = \frac{(-\mathrm{i})^\ell}{2\pi n k_0}\int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot\mathbf{r}}S_{\ell m}(\pm\cos\bar{\theta}, 2\pi - \bar{\phi}) \tag{26}$$

and with Eq. (23) this becomes

$$\Pi_{\ell m}(r, \theta, -\phi) = (-1)^m\frac{(-\mathrm{i})^\ell}{2\pi n k_0} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot\mathbf{r}}S_{\ell,-m}(\pm\cos\bar{\theta}, \bar{\phi} - 2\pi). \tag{27}$$

Here we can replace $\bar{\phi} - 2\pi$ by $\bar{\phi}$ since $S_{\ell m}(z,\alpha)$ is periodic in α with period 2π. Then the right-hand side of Eq. (27) is just $(-1)^m\Pi_{\ell,-m}(r,\theta,\phi)$, provided we represent $\Pi_{\ell m}(r,\theta,\phi)$ for $m < 0$ also by Eq. (22).

5. Vector multipole fields

The vector multipole fields, indicated by $\mathbf{A}_{\eta\ell m}(\mathbf{r})$, in a medium with index of refraction n are solutions of the vector Helmholz equation

$$(\nabla^2 + n^2k_0^2)\mathbf{A}_{\eta\ell m}(\mathbf{r}) = 0. \tag{28}$$

The subscript $\eta = \pm 1$ distinguishes between two possible types of fields. We consider divergence-free (transverse) solutions only:

$$\nabla\cdot\mathbf{A}_{\eta\ell m}(\mathbf{r}) = 0. \tag{29}$$

There also exists a longitudinal (vanishing curl) solution, which could be indicated by $\eta = 0$. The standard solutions, which form a complete transverse set and are normalized on the unit sphere, are given by [23]

$$A_{1\ell m}(r) = \frac{1}{\sqrt{\ell(\ell+1)}} L\Pi_{\ell m}(r), \qquad (30)$$

$$A_{-1\ell m}(r) = -\frac{\mathrm{i}}{nk_0}\frac{1}{\sqrt{\ell(\ell+1)}} \nabla \times (L\Pi_{\ell m}(r)) \qquad (31)$$

for $\ell = 1, 2, \ldots$ and $m = -\ell, \ldots, \ell$ (there is no transverse vector multipole for $\ell = 0$), and $L = -\mathrm{i}r \times \nabla$. From the above we readily verify the reciprocity relations between the $\eta = 1$ and $\eta = -1$ fields

$$\nabla \times A_{\eta\ell m}(r) = \mathrm{i}nk_0\eta A_{-\eta\ell m}(r). \qquad (32)$$

The vector multipole fields are expressed in terms of the scalar multipole fields, for which we have an angular spectrum representation, but the appearance of the operator L is cumbersome. For the $\eta = 1$ fields we shall use the alternative form [24,25]

$$A_{1\ell m}(r) = h_\ell^{(1)}(nk_0 r) T_{\ell\ell m}(\theta, \phi), \qquad (33)$$

in terms of vector spherical harmonics $T_{\ell\ell m}(\theta, \phi)$, which are in general defined as

$$T_{j\ell m}(\theta, \phi) = \sum_{\mu'\mu} (\ell\mu' 1\mu | jm) Y_{\ell\mu'}(\theta, \phi) e_\mu, \qquad (34)$$

with $(\ell\mu' 1\mu | jm)$ Clebsch–Gordan coefficients and e_μ spherical unit vectors. The summation on the right-hand side has at most three terms since $\mu = 1, 0, -1$, and the Clebsch–Gordan coefficients vanish unless $\mu' = m - \mu$. Then the $\eta = 1$ fields can then be written as

$$A_{1\ell m}(r) = \sum_\mu (\ell m - \mu 1\mu | \ell m) \Pi_{\ell, m-\mu}(\theta, \phi) e_\mu. \qquad (35)$$

In the representation of Eq. (35) we can substitute the angular spectrum representation of the scalar multipole fields, given by Eq. (22). We then obtain

$$A_{1\ell m}(r) = \frac{(-\mathrm{i})^\ell}{2\pi nk_0} \int \mathrm{d}^2 k_\parallel \frac{1}{\beta} \mathrm{e}^{\mathrm{i}K_\pm \cdot r} V_{\ell m}(\pm\cos\bar{\theta}, \bar{\phi}), \qquad (36)$$

in terms of the generalized vector spherical harmonics, defined by

$$V_{\ell m}(z, \alpha) = \sum_\mu (\ell m - \mu 1\mu | \ell m) S_{\ell, m-\mu}(z, \alpha) e_\mu, \qquad (37)$$

with $S_{\ell m}(z, \alpha)$ the generalized spherical harmonics from Eq. (18). We only need the generalization of $T_{j\ell m}(\theta, \phi)$ for $j = \ell$, so we have left out a redundant subscript ℓ in the notation. The angular spectrum representation for the $\eta = -1$ fields now follows from Eq. (32) with $\eta = 1$:

$$A_{-1\ell m}(r) = \frac{(-\mathrm{i})^\ell}{2\pi nk_0} \int \mathrm{d}^2 k_\parallel \frac{1}{\beta} \mathrm{e}^{\mathrm{i}K_\pm \cdot r} \hat{K}_\pm \times V_{\ell m}(\pm\cos\bar{\theta}, \bar{\phi}), \qquad (38)$$

where $\hat{K}_\pm = K_\pm/(nk_0)$.

6. Electromagnetic multipole fields

The electric and magnetic fields, $E(r)$ and $B(r)$, emitted by a localized source near the origin of coordinates, are a solution of the free-space Maxwell equations. Since the vector multipole fields from the previous section form a complete set of transverse functions, we can express the electric and magnetic fields as superpositions of the vector multipole fields. For the electric field we write

$$E(r) = \frac{\mathrm{i}k_0^3}{4\pi\varepsilon_0} \sum_{\eta\ell m} b_{\eta\ell m} A_{\eta\ell m}(r). \qquad (39)$$

The magnetic field then follows from a Maxwell equation according to

$$B(r) = -\frac{\mathrm{i}}{\omega} \nabla \times E(r) \qquad (40)$$

and with Eq. (32) this is explicitly

$$B(r) = \frac{n}{c}\frac{\mathrm{i}k_0^3}{4\pi\varepsilon_0} \sum_{\eta\ell m} \eta b_{\eta\ell m} A_{-\eta\ell m}(r). \qquad (41)$$

A pure (η, ℓ, m) multipole field has only one term in both Eqs. (39) and (41). A multipole field with $\eta = -1$ $(\eta = 1)$ is commonly referred to as an electric (a magnetic) multipole field. The constants $b_{\eta\ell m}$ in Eqs. (39) and (41) are called the multipole coefficients.

The multipole coefficients $b_{\eta\ell m}$ are determined by the source of the radiation. Explicit expressions can be found in Ref. [23] for fields in vacuum. We now include the possibility of an embedding medium with index of refraction n, and we consider the case where the source is entirely accounted for by a given current density $\boldsymbol{j}(\boldsymbol{r})$ with dimensions much smaller than the wavelength of the radiation. Then the multipole coefficients can be obtained as

$$b_{-1\ell m} = \frac{\mathrm{i}}{\omega}\frac{4\pi}{k_0}\frac{(nk_0)^{\ell}}{(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{j}(\boldsymbol{r})\cdot\nabla \mathsf{Y}_{\ell m}(\boldsymbol{r})^*, \tag{42}$$

$$b_{1\ell m} = \frac{n}{c}\frac{4\pi}{k_0}\frac{(nk_0)^{\ell}}{(2\ell+1)!!}\frac{1}{\sqrt{\ell(\ell+1)}} \times \int \mathrm{d}^3\boldsymbol{r}[\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})]\cdot\nabla \mathsf{Y}_{\ell m}(\boldsymbol{r})^*. \tag{43}$$

These involve the solid spherical harmonics $\mathsf{Y}_{\ell m}(\boldsymbol{r})$, defined as

$$\mathsf{Y}_{\ell m}(\boldsymbol{r}) = r^{\ell} Y_{\ell m}(\theta,\phi) \tag{44}$$

and the gradient of these functions is

$$\nabla \mathsf{Y}_{\ell m}(\boldsymbol{r}) = \sqrt{\ell(2\ell+1)} \times \sum_{\mu}(\ell-1m-\mu 1\mu|\ell m)\mathsf{Y}_{\ell-1,m-\mu}(\boldsymbol{r})\boldsymbol{e}_{\mu}. \tag{45}$$

It is interesting to see that the multipole coefficients depend on the index of refraction of the embedding medium. They are proportional to n^{ℓ} for an electric multipole and to $n^{\ell+1}$ for a magnetic multipole.

7. Sources of multipole radiation

The multipole fields are uniquely determined by a given current density $\boldsymbol{j}(\boldsymbol{r})$ of the (monochromatic) source. We shall first consider a point source at the origin of coordinates, with a current density given by

$$\boldsymbol{j}(\boldsymbol{r}) = -\mathrm{i}\omega\boldsymbol{d}\delta(\boldsymbol{r}), \tag{46}$$

where $\boldsymbol{d}$ is an arbitrary vector. The multipole coefficients $b_{\eta\ell m}$ for this current distribution can be found by evaluating the integrals in Eqs. (42) and (43), and using expression (45) for the gradient of the solid spherical harmonics. Due to the delta function in Eq. (46), the only contribution comes from the point $\boldsymbol{r}=0$. However, the solid spherical harmonics are homogeneous polynomials of degree ℓ in x, y and z, and they vanish at the origin unless $\ell=m=0$. To be specific,

$$\mathsf{Y}_{\ell m}(0) = \frac{1}{\sqrt{4\pi}}\delta_{\ell,0}\delta_{m,0}. \tag{47}$$

Therefore, the sum on the right-hand side of Eq. (45) only gives a contribution if $\ell=1$ and $\mu=m$. We find for the multipole coefficients

$$b_{\eta\ell m} = 2n\sqrt{\frac{2\pi}{3}}\delta_{\eta,-1}\delta_{\ell,1}\boldsymbol{d}\cdot\boldsymbol{e}_m^*, \tag{48}$$

where we have used $\boldsymbol{r}\times[\boldsymbol{d}\delta(\boldsymbol{r})]=0$. We conclude that the current distribution (46) represents an electric ($\eta=-1$) dipole ($\ell=1$), and it gives a contribution to the three m values depending on $\boldsymbol{d}$. To obtain electric dipole radiation that only contains one specific value of m, the dipole moment $\boldsymbol{d}$ must have the form

$$\boldsymbol{d} = \bar{d}\boldsymbol{e}_m, \tag{49}$$

with $\bar{d}$ an arbitrary complex number, e.g., $\boldsymbol{d}$ must be proportional to the spherical unit vector $\boldsymbol{e}_m$. For $m=0$ this is a linear dipole along the z-axis, since $\boldsymbol{e}_0=\boldsymbol{e}_z$. Since we consider monochromatic fields, all quantities are complex amplitudes and the corresponding time dependent quantities are, for instance, $\boldsymbol{d}(t)=\mathrm{Re}[\boldsymbol{d}\exp(-\mathrm{i}\omega t)]$. With the definition of the spherical unit vectors for $m=\pm1$

$$\boldsymbol{e}_{\pm1} = \frac{1}{\sqrt{2}}(\mp\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y), \tag{50}$$

we then see that $m=1$ ($m=-1$) corresponds to a dipole moment $\boldsymbol{d}(t)$ that rotates counterclockwise (clockwise) in the xy-plane with constant amplitude.

As a second example we consider

$$\boldsymbol{j}(\boldsymbol{r}) = -\boldsymbol{p}\times\nabla\delta(\boldsymbol{r}), \tag{51}$$

with $\boldsymbol{p}$ an arbitrary vector. In order to remove the derivatives of the delta function, we integrate by parts first. This effectively moves the derivatives to the factors $\mathsf{Y}_{\ell-1,m-\mu}(\boldsymbol{r})$ on the right-hand side of Eq. (45), and these derivatives can be evaluated with the same formula. For the multipole coefficients we then find

$$b_{\eta\ell m} = \frac{2n^2}{c}\sqrt{\frac{2\pi}{3}}\delta_{\eta,1}\delta_{\ell,1}\boldsymbol{p}\cdot\boldsymbol{e}_m^*. \tag{52}$$

Evidently, this is magnetic dipole radiation from a magnetic dipole moment $\boldsymbol{p}$. For an arbitrary $\boldsymbol{p}$, all three m values will occur in the radiation field, and we get a pure multipole with one m value if

$$\boldsymbol{p} = \bar{p}\boldsymbol{e}_m, \tag{53}$$

just like for the electric dipole.

An interesting method for obtaining the current distributions of the higher-order multipoles is given by van Bladel [26]. The next leading term has the form

$$\boldsymbol{j}(\boldsymbol{r}) = \frac{1}{6}\mathrm{i}\omega[\nabla\delta(\boldsymbol{r})]\cdot\overleftrightarrow{Q}, \tag{54}$$

with $\overleftrightarrow{Q}$ an arbitrary symmetric second-order Cartesian tensor, the electric quadrupole tensor. Upon evaluating the integral in Eq. (43) we find that the result is proportional to $\sum_{\alpha\beta}Q_{\alpha\beta}\boldsymbol{e}_\beta\times\boldsymbol{e}_\alpha$ with $\alpha, \beta = x, y, z$, and this is zero since Q is symmetric. Therefore, $b_{1\ell m} = 0$. For the electric multipole coefficients we obtain

$$b_{-1\ell m} = \frac{1}{3}n^2k_0\sqrt{\frac{\pi}{5}}\delta_{\ell,2}\sum_{\mu\mu'}(1\mu1\mu'|2m)\boldsymbol{e}_\mu^*\cdot\overleftrightarrow{Q}\cdot\boldsymbol{e}_{\mu'}^*. \tag{55}$$

Clearly, $\boldsymbol{j}(\boldsymbol{r})$ in Eq. (54) gives electric quadrupole ($\ell = 2$) radiation, and for an arbitrary quadrupole tensor $\overleftrightarrow{Q}$ all m values will contribute. The question comes up what the form of $\overleftrightarrow{Q}$ must be so that only one m value survives, in analogy to Eqs. (49) and (53) for the electric and magnetic dipoles, respectively. The answer is

$$\overleftrightarrow{Q} = \overline{Q}\overleftrightarrow{E}_m^{(2)}, \tag{56}$$

where $\overleftrightarrow{E}_m^{(2)}$ are "spherical unit tensors", just like the spherical unit vectors. They are defined as (see Appendix A for the relation to the irreducible tensor components of $\overleftrightarrow{Q}$):

$$\overleftrightarrow{E}_q^{(k)} = \sum_{\tau\tau'}(1\tau1\tau'|kq)\boldsymbol{e}_\tau\boldsymbol{e}_{\tau'} \tag{57}$$

for $k = 0$, 1 and 2, $q = -k,\ldots,k$, and they form a complete set for second-rank tensors. From $\boldsymbol{e}_\mu^*\cdot\boldsymbol{e}_\tau = \delta_{\mu\tau}$ and the orthonormality of the Clebsch–Gordan coefficients we then have

$$\sum_{\mu\mu'}(1\mu1\mu'|2m)\boldsymbol{e}_\mu^*\cdot\overleftrightarrow{E}_q^{(k)}\cdot\boldsymbol{e}_{\mu'}^* = \delta_{k,2}\delta_{q,m}, \tag{58}$$

showing that the operation on the left, which is the same as in Eq. (55), just filters out the $k = 2$ and $q = m$ value.

8. Properties of the generalized vector spherical harmonics

If we replace z by $-z$ in the definition (37) of the generalized vector spherical harmonics, and use Eq. (20), we find

$$\begin{aligned}\boldsymbol{V}_{\ell m}(-z,\alpha) &= (-1)^{\ell+m+1}\sum_{\mu=\pm1}(\ell m-\mu1\mu|\ell m)S_{\ell,m-\mu}(z,\alpha)\boldsymbol{e}_\mu \\ &\quad + (-1)^{\ell+m}(\ell m10|\ell m)S_{\ell,m}(z,\alpha)\boldsymbol{e}_z.\end{aligned} \tag{59}$$

If we then combine this with $\boldsymbol{V}_{\ell m}(z,\alpha)$ in two different ways we obtain

$$\begin{aligned}&\boldsymbol{V}_{\ell m}(z,\alpha) + (-1)^{\ell+m}\boldsymbol{V}_{\ell m}(-z,\alpha) \\ &\quad = 2(\ell m10|\ell m)S_{\ell,m}(z,\alpha)\boldsymbol{e}_z,\end{aligned} \tag{60}$$

$$\begin{aligned}&\boldsymbol{V}_{\ell m}(z,\alpha) - (-1)^{\ell+m}\boldsymbol{V}_{\ell m}(-z,\alpha) \\ &\quad = 2\sum_{\mu=\pm1}(\ell m-\mu1\mu|\ell m)S_{\ell,m-\mu}(z,\alpha)\boldsymbol{e}_\mu.\end{aligned} \tag{61}$$

Here we notice that the right-hand sides of Eqs. (60) and (61) are perpendicular and parallel to the xy-plane, respectively. We shall use this in Section 10 to verify boundary conditions.

From definition (18) we observe

$$S_{\ell m}(z,\alpha) = \mathrm{e}^{\mathrm{i}m\alpha}S_{\ell m}(z,0). \tag{62}$$

The α-dependence of each term in $\boldsymbol{V}_{\ell m}(z,\alpha)$ in Eq. (37) is therefore exp[i$(m-\mu)\alpha$]. When considering $\boldsymbol{V}_{\ell m}(z,\alpha)^*$, each term has a factor exp[-i$(m-\mu)\alpha$], and therefore

$$\boldsymbol{V}_{\ell m}(z,\alpha)\cdot\boldsymbol{V}_{\ell m}(z,\alpha)^* = \boldsymbol{V}_{\ell m}(z,0)\cdot\boldsymbol{V}_{\ell m}(z,0)^*, \tag{63}$$

since the cross terms cancel due to $\boldsymbol{e}_\mu\cdot\boldsymbol{e}_{\mu'}^* = \delta_{\mu\mu'}$.

Now we consider the relation between positive and negative m values, for the case $\alpha = 0$. From Eq. (50) we see that $\boldsymbol{e}_x\cdot\boldsymbol{e}_\mu = -\mu/\sqrt{2}$, from which

$$e_x \cdot V_{\ell m}(z,0) = -\frac{1}{\sqrt{2}} \sum_{\mu=\pm 1} (\ell m - \mu 1 \mu | \ell m) \mu S_{\ell,m-\mu}(z,0). \tag{64}$$

Then we replace m by $-m$ and use the property $(\ell - m - \mu 1 \mu | \ell - m) = -(\ell m + \mu 1 - \mu | \ell m)$. From Eq. (23) we have $S_{\ell,-m}(z,0) = (-1)^m S_{\ell m}(z,0)$, which then yields

$$e_x \cdot V_{\ell,-m}(z,0) = (-1)^{m+1} e_x \cdot V_{\ell m}(z,0). \tag{65}$$

Along similar lines we find

$$e_y \cdot V_{\ell,-m}(z,0) = (-1)^m e_y \cdot V_{\ell m}(z,0), \tag{66}$$

$$e_z \cdot V_{\ell,-m}(z,0) = (-1)^{m+1} e_z \cdot V_{\ell m}(z,0). \tag{67}$$

Together with Eq. (63) this shows that $V_{\ell m}(z,\alpha) \cdot V_{\ell m}(z,\alpha)^*$ is independent of α and the sign of m.

From definition (37) we immediately see

$$e_z \cdot V_{\ell m}(z,\alpha) = (\ell m 1 0 | \ell m) S_{\ell m}(z,\alpha). \tag{68}$$

From Eq. (50) we find

$$(e_x \cos\alpha + e_y \sin\alpha) \cdot e_\mu = -\frac{\mu}{\sqrt{2}} e^{i\mu\alpha} \tag{69}$$

and therefore

$$(e_x \cos\alpha + e_y \sin\alpha) \cdot V_{\ell m}(z,\alpha) = -\frac{1}{\sqrt{2}} \sum_{\mu=\pm 1} (\ell m - \mu 1 \mu | \ell m) \mu e^{i\mu\alpha} S_{\ell,m-\mu}(z,\alpha). \tag{70}$$

The three Clebsch–Gordan coefficients that appear in the definition (37) of $V_{\ell m}(z,\alpha)$ are explicitly

$$(\ell m - 1 1 1 | \ell m) = -\sqrt{\frac{(\ell - m + 1)(\ell + m)}{2\ell(\ell+1)}}, \tag{71}$$

$$(\ell m 1 0 | \ell m) = \frac{m}{\sqrt{\ell(\ell+1)}}, \tag{72}$$

$$(\ell m + 1 1 - 1 | \ell m) = \sqrt{\frac{(\ell + m + 1)(\ell - m)}{2\ell(\ell+1)}}. \tag{73}$$

We then notice that the numerators are just the factors in the recursion relation (21) for the generalized spherical harmonics. Then we set $z = \cos\beta$ in Eq. (21), with again the understanding that $\cos\beta$ can be complex, and $\sin\beta = (1 - z^2)^{1/2}$. This gives the alternative form

$$\frac{1}{\sqrt{2}} \sin\beta \sum_{\mu=\pm 1} (\ell m - \mu 1 \mu | \ell m) \mu e^{i\mu\alpha} S_{\ell,m-\mu}(\cos\beta,\alpha) = \cos\beta (\ell m 1 0 | \ell m) S_{\ell m}(\cos\beta,\alpha). \tag{74}$$

The summation on the left-hand side is the same as in Eq. (70), and the right-hand side equals $\cos\beta$ times the right-hand side of Eq. (68). In this way, Eq. (74) becomes

$$[\sin\beta(e_x \cos\alpha + e_y \sin\alpha) + e_z \cos\beta] \cdot V_{\ell m}(\cos\beta,\alpha) = 0. \tag{75}$$

For $(\beta,\alpha) = (\theta,\phi)$ the term in square brackets is the radial unit vector $\hat{r}$, and $V_{\ell m}(\cos\theta,\phi) = T_{\ell\ell m}(\theta,\phi)$. Therefore, Eq. (75) generalizes the theorem

$$\hat{r} \cdot T_{\ell\ell m}(\theta,\phi) = 0, \tag{76}$$

which expresses that the magnetic multipole fields $A_{1\ell m}(r)$, Eq. (33), are transverse in the sense that they are perpendicular to r for all points in space.

9. Multipole near a perfect conductor

The angular spectrum representation of the electromagnetic multipole fields is particularly useful for the study of reflection and refraction by an interface. As an example we consider a pure (η, ℓ, m) multipole located on the z-axis and a distance H above the xy-plane, as illustrated in Fig. 2. The region $z > 0$ is occupied by a dielectric material with index of refraction n, with the multipole embedded in it, and the half-space $z < 0$ is assumed to be a perfect conductor. In the perfect conductor we have $E(r) = 0$ and $B(r) = 0$, and therefore the boundary conditions are that $E(r)_\parallel$ and $B(r)_\perp$ vanish at $z = 0^+$. In other words, $E(r)$ must be along the z-axis and $B(r)$ must be in the xy-plane at $z = 0^+$.

In the region $z > 0$ the electric field can be written as a superposition of the source (s) field and the reflected (r) field:

$$E(r) = E_s(r) + E_r(r) \tag{77}$$

and similarly for the magnetic field. The angular spectrum representations of $E(r)$ and $B(r)$ of a multipole, the source field, are given by Eqs. (39) and (41), by retaining only one term, and with expression (36) or (38) substituted for the vector multipole field. This representation is, however, the

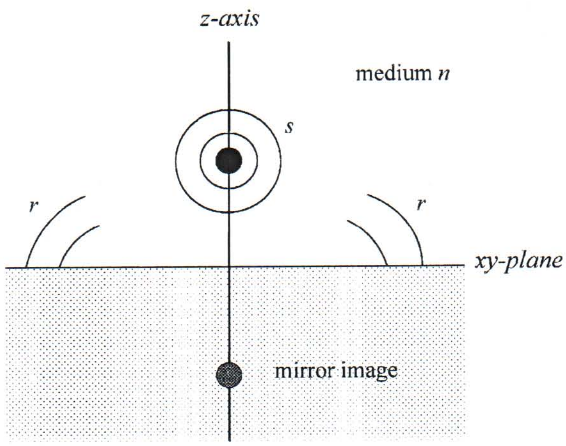

Fig. 2. The multipole source is located on the z-axis, a distance H above the xy-plane, and embedded in a material with index of refraction n. The region $z<0$ is a perfect conductor. The mirror image of the multipole is a located on the z-axis, a distance H below the xy-plane.

field of a multipole located at the origin of coordinates. An advantage of the angular spectrum is that the $\mathbf{r}$ dependence of the field only enters as exp $(\mathrm{i}\mathbf{K}_\pm\cdot\mathbf{r})$ in the integrand, and we can easily shift the location of the multipole to $H\mathbf{e}_z$ by the substitution $\mathbf{r}\rightarrow\mathbf{r}-H\mathbf{e}_z$. We then obtain for the electric and magnetic source fields of an electric multipole

$$\mathbf{E}_s(\mathbf{r}) = -\frac{(-\mathrm{i})^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 n}b_{-1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot(\mathbf{r}-H\mathbf{e}_z)}\hat{\mathbf{K}}_\pm\times\mathbf{V}_{\ell m}(\pm\cos\bar{\theta},\bar{\phi}), \tag{78}$$

$$\mathbf{B}_s(\mathbf{r}) = \frac{(-\mathrm{i})^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 c}b_{-1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot(\mathbf{r}-H\mathbf{e}_z)}\mathbf{V}_{\ell m}(\pm\cos\bar{\theta},\bar{\phi}) \tag{79}$$

and the fields of a magnetic multipole are

$$\mathbf{E}_s(\mathbf{r}) = -\frac{(-\mathrm{i})^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 n}b_{1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot(\mathbf{r}-H\mathbf{e}_z)}\mathbf{V}_{\ell m}(\pm\cos\bar{\theta},\bar{\phi}), \tag{80}$$

$$\mathbf{B}_s(\mathbf{r}) = -\frac{(-\mathrm{i})^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 c}b_{1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_\pm\cdot(\mathbf{r}-H\mathbf{e}_z)}\hat{\mathbf{K}}_\pm\times\mathbf{V}_{\ell m}(\pm\cos\bar{\theta},\bar{\phi}). \tag{81}$$

The upper and lower signs now refer to $z>H$ and $0<z<H$, respectively.

The reflected fields have to be determined such that the sum of the s and r fields satisfy the boundary conditions at the interface. We shall employ the method of images, which consists of guessing the answer and then proving afterwards that it is correct. For a given (η,ℓ,m), the reflected field is again a multipole field with the same (η,ℓ,m) and the same $b_{\eta\ell m}$, but the location of the source is at the mirror position $-H\mathbf{e}_z$. We then only need the solution with the upper sign. In addition, the field picks up a phase factor of $(-1)^{\ell+m+1}$ for an electric multipole and $(-1)^{\ell+m}$ for a magnetic multipole. The reflected electric and magnetic fields of an electric multipole are

$$\mathbf{E}_r(\mathbf{r}) = (-1)^{m+1}\frac{\mathrm{i}^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 n}b_{-1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_+\cdot(\mathbf{r}+H\mathbf{e}_z)}\hat{\mathbf{K}}_+\times\mathbf{V}_{\ell m}(\cos\bar{\theta},\bar{\phi}), \tag{82}$$

$$\mathbf{B}_r(\mathbf{r}) = (-1)^{m}\frac{\mathrm{i}^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 c}b_{-1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_+\cdot(\mathbf{r}+H\mathbf{e}_z)}\mathbf{V}_{\ell m}(\cos\bar{\theta},\bar{\phi}) \tag{83}$$

and for a magnetic multipole we have

$$\mathbf{E}_r(\mathbf{r}) = (-1)^{m}\frac{\mathrm{i}^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 n}b_{1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_+\cdot(\mathbf{r}+H\mathbf{e}_z)}\mathbf{V}_{\ell m}(\cos\bar{\theta},\bar{\phi}), \tag{84}$$

$$\mathbf{B}_r(\mathbf{r}) = (-1)^{m}\frac{\mathrm{i}^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 c}b_{1\ell m} \times \int \mathrm{d}^2\mathbf{k}_\parallel \frac{1}{\beta}\mathrm{e}^{\mathrm{i}\mathbf{K}_+\cdot(\mathbf{r}+H\mathbf{e}_z)}\hat{\mathbf{K}}_+\times\mathbf{V}_{\ell m}(\cos\bar{\theta},\bar{\phi}). \tag{85}$$

10. Verification of the boundary conditions

Since the reflected fields are again multipole fields for a medium with index of refraction n, with the source below the surface, they satisfy Maxwell's equations in $z>0$. Therefore we only have to verify that in $z=0^+$ the total electric field is perpendicular to the surface and the total magnetic field is parallel to the surface. For the source field we need the solution with the lower sign. With the definition (1) of $\boldsymbol{K}_\pm$ we have $\boldsymbol{K}_\pm \cdot (\boldsymbol{r} \pm H\boldsymbol{e}_z) = \boldsymbol{k}_\parallel \cdot \boldsymbol{r} \pm \beta z + \beta H$, and the expression with the upper (lower) sign appears in the exponential in the reflected (source) field in the region $0 < z < H$. For $z = 0^+$, both expressions are the same, and we can easily add the reflected field and the source field. For the electric multipole we find the fields at $z = 0^+$ to be

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{(-\mathrm{i})^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 n} b_{-1\ell m} \times \int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\parallel\cdot\boldsymbol{r}+\beta H)} \times [\hat{\boldsymbol{K}}_- \times \boldsymbol{V}_{\ell m}(-\cos\bar{\theta}, \bar{\phi}) - (-1)^{\ell+m}\hat{\boldsymbol{K}}_+ \times \boldsymbol{V}_{\ell m}(\cos\bar{\theta}, \bar{\phi})], \tag{86}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{(-\mathrm{i})^{\ell+1}k_0^2}{8\pi^2\varepsilon_0 c} b_{-1\ell m} \times \int \mathrm{d}^2\boldsymbol{k}_\parallel \frac{1}{\beta} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\parallel\cdot\boldsymbol{r}+\beta H)} \times [\boldsymbol{V}_{\ell m}(-\cos\bar{\theta}, \bar{\phi}) - (-1)^{\ell+m}\boldsymbol{V}_{\ell m}(\cos\bar{\theta}, \bar{\phi})] \tag{87}$$

and similar expressions hold for the magnetic multipole. The term in square brackets in Eq. (87) is the same as the left-hand side of Eq. (61) with $z = -\cos\bar{\theta}$, $\alpha = \bar{\phi}$, and therefore we see immediately that $\boldsymbol{B}(\boldsymbol{r})$ is in the xy-plane. The term in square brackets in Eq. (86) is first split as

$$\begin{aligned}&\hat{\boldsymbol{K}}_- \times \boldsymbol{V}_{\ell m}(-\cos\bar{\theta}, \bar{\phi}) - (-1)^{\ell+m}\hat{\boldsymbol{K}}_+ \times \boldsymbol{V}_{\ell m}(\cos\bar{\theta}, \bar{\phi})\\ &= \frac{1}{nk_0}\boldsymbol{k}_\parallel \times [\boldsymbol{V}_{\ell m}(-\cos\bar{\theta}, \bar{\phi}) - (-1)^{\ell+m}\boldsymbol{V}_{\ell m}(\cos\bar{\theta}, \bar{\phi})] - \frac{\beta}{nk_0}\boldsymbol{e}_z\\ &\quad \times [\boldsymbol{V}_{\ell m}(-\cos\bar{\theta}, \bar{\phi}) + (-1)^{\ell+m}\boldsymbol{V}_{\ell m}(\cos\bar{\theta}, \bar{\phi})].\end{aligned} \tag{88}$$

From Eq. (60) we see that the second expression in square brackets is proportional to $\boldsymbol{e}_z$, giving $\boldsymbol{e}_z \times \boldsymbol{e}_z = 0$. With Eq. (61) we find that the first expression in square brackets is a combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_{-1}$, and with

$$\boldsymbol{k}_\parallel \times \boldsymbol{e}_\mu = -\frac{\mathrm{i}}{\sqrt{2}} k_\parallel \mathrm{e}^{\mathrm{i}\mu\bar{\phi}} \boldsymbol{e}_z, \quad \mu = \pm 1, \tag{89}$$

as follows from Eqs. (9) and (50), we find that the right-hand side of Eq. (88) is proportional to $\boldsymbol{e}_z$. This proves that the sum of the source and the image fields for an electric multipole satisfy the boundary conditions, and the same can be verified for the fields of the magnetic multipole along the same lines.

11. Image sources

The reflected field of an (η, ℓ, m) multipole is the same as the source field itself, except that it originates from the image position, and in addition it picks up a possible minus sign. The information about the source is contained entirely in the multipole coefficients $b_{\eta\ell m}$, so effectively a reflected field follows from the replacement

$$b_{-1\ell m} \to (-1)^{\ell+m+1} b_{-1\ell m}, \tag{90}$$

$$b_{1\ell m} \to (-1)^{\ell+m} b_{1\ell m} \tag{91}$$

in the source field. But from Eqs. (42) and (43) we see that the multipole coefficients are determined by the current density $\boldsymbol{j}(\boldsymbol{r})$, so we could equally assign the phase factors to the corresponding $\boldsymbol{j}(\boldsymbol{r})$. For an electric dipole we have $\boldsymbol{j}(\boldsymbol{r}) = -\mathrm{i}\omega\boldsymbol{d}\delta(\boldsymbol{r})$, so the phase factor effectively changes the dipole moment $\boldsymbol{d}$. When we decompose $\boldsymbol{d}$ in spherical unit vectors as in Eq. (A.1):

$$\boldsymbol{d} = \sum_q d_q \boldsymbol{e}_q^*, \tag{92}$$

which can also be written as

$$\boldsymbol{d} = \sum_q (-1)^q d_{-q} \boldsymbol{e}_q, \tag{93}$$

then each term q is the source of electric dipole radiation with $m = q$, according to Eq. (49). For the image source, the term with q then picks up a factor of $(-1)^{\ell+m+1}$ with $\ell = 1$ and $m = q$, which is $(-1)^q$. By superposition, the image dipole of $\boldsymbol{d}$ given by Eq. (93) is then

$$\boldsymbol{d}^{(im)} = \sum_q d_{-q} \boldsymbol{e}_q, \tag{94}$$

which is the same as

$$\boldsymbol{d}^{(im)} = \sum_q (-1)^q d_q \boldsymbol{e}_q^*. \tag{95}$$

Comparison with Eq. (92) then shows that the components with $q = \pm 1$ get a minus sign and the $q = 0$ component remains unaltered. But then, the vectors $\mathbf{e}_\pm$ are in the xy-plane whereas $\mathbf{e}_0$ is along the z-axis, so if we write $\mathbf{d} = \mathbf{d}_\perp + \mathbf{d}_\parallel$ then the image dipole moment is

$$\mathbf{d}^{(im)} = \mathbf{d}_\perp - \mathbf{d}_\parallel. \tag{96}$$

The same holds for the magnetic dipole $\mathbf{p}$, except that each spherical component p_q acquires an extra minus sign, as compared to the electric dipole. Therefore the image of a magnetic dipole moment is

$$\mathbf{p}^{(im)} = \mathbf{p}_\parallel - \mathbf{p}_\perp. \tag{97}$$

Let us now consider the electric quadrupole $\overleftrightarrow{Q}$. According to Eq. (A.10), $\overleftrightarrow{Q}$ can be decomposed in spherical unit tensors according to

$$\overleftrightarrow{Q} = \sum_{kq} Q_q^{(k)} \overleftrightarrow{E}_q^{(k)*} \tag{98}$$

and since $\overleftrightarrow{Q}$ is a symmetric Cartesian tensor, the components with $k = 1$ vanish. We can also write $\overleftrightarrow{Q}$ as, Eq. (A.12),

$$\overleftrightarrow{Q} = \sum_{kq} (-1)^{k+q} Q_{-q}^{(k)} \overleftrightarrow{E}_q^{(k)} \tag{99}$$

in analogy to Eq. (93) for the electric dipole. As shown in Section 7, an electric quadrupole field with a given m is emitted by a quadrupole moment proportional to $\overleftrightarrow{E}_m^{(2)}$. For the mirror source, a factor $(-1)^{\ell+m+1}$ with $\ell = 2$ appears, which is $(-1)^{q+1}$ for each term in Eq. (99). This gives the mirror quadrupole

$$\overleftrightarrow{Q}^{(im)} = -\sum_q Q_{-q}^{(2)} \overleftrightarrow{E}_q^{(2)} \tag{100}$$

and this is

$$\overleftrightarrow{Q}^{(im)} = \sum_q (-1)^{q+1} Q_q^{(2)} \overleftrightarrow{E}_q^{(2)*}. \tag{101}$$

As compared to Eq. (98), we see that each term with $k = 2$ picks up a factor $(-1)^{q+1}$. The mirror image does not have the $k = 0$ contribution, as does $\overleftrightarrow{Q}$, although it could be added since it would not contribute to the image field anyway, according to Eqs. (55) and (58).

12. Intensity distribution

In Section 9 we obtained angular spectrum representations for the electric and magnetic source and reflected fields in the half-space $z > 0$. Of particular interest are the far fields $r \to \infty$, and the radiation pattern (intensity distribution) of a multipole near the perfect conductor. In the far field we have $z > H$, so that only the upper signs are used for the source fields. Then each partial wave in the source and reflected fields has wave vector $\mathbf{K}_+$. An asymptotic approximation for r large of an angular spectrum can be made with the method of stationary phase [27,28]. For an arbitrary function $f(\mathbf{k}_\parallel)$ we have

$$\int d^2 k_\parallel \frac{1}{\beta} e^{i\mathbf{K}_+\cdot\mathbf{r}} f(\mathbf{k}_\parallel) \approx -\frac{2\pi i}{r} e^{ink_0 r} f(\mathbf{k}_{\parallel,0}), \tag{102}$$

where $\mathbf{k}_{\parallel,0}$ is the stationary point in the $\mathbf{k}_\parallel$-plane. This point is given by

$$\mathbf{k}_{\parallel,0} = nk_0 \sin\theta(\mathbf{e}_x \cos\phi + \mathbf{e}_y \sin\phi), \tag{103}$$

for a given observation direction (θ, ϕ). Angle $\bar{\phi}$ in the angular spectra is the angle of $\mathbf{k}_\parallel$ in the $\mathbf{k}_\parallel$-plane, according to Eq. (9), and comparison with Eq. (104) then shows that in the stationary point we have $(\bar{\phi})_0 = \phi$. From Eq. (103) it follows that the magnitude of $\mathbf{k}_{\parallel,0}$ equals $nk_0\sin\theta$, which gives $(\sin\bar{\theta})_0 = \sin\theta$ with Eq. (10), and from Eqs. (2) and (11) we then have $(\cos\bar{\theta})_0 = \cos\theta$. From Eq. (12) and $\hat{\mathbf{K}}_\pm = \mathbf{K}_\pm/(nk_0)$ we obtain $(\hat{\mathbf{K}}_+)_0 = \hat{\mathbf{r}}$. The phase factors involving the distance H become $\{\exp[i\mathbf{K}_+\cdot(\pm H\mathbf{e}_z)]\}_0 = \exp(\pm inh\cos\theta)$, where we have set $h = k_0 H$ for the dimensionless distance between the multipole and the interface. We then find the asymptotic approximation for the total electric field of an electric multipole to be (we shall write an equal sign instead of $\approx$)

$$\mathbf{E}(\mathbf{r}) = \frac{(-i)^\ell k_0^2}{4\pi\varepsilon_0 nr} b_{-1\ell m} e^{in(k_0 r - h\cos\theta)} \times [1 - (-1)^{\ell+m} e^{2inh\cos\theta}]\hat{\mathbf{r}} \times \mathbf{V}_{\ell m}(\cos\theta, \phi), \tag{104}$$

and for a magnetic multipole we obtain

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{(-\mathrm{i})^{\ell} k_0^2}{4\pi\varepsilon_0 n r} b_{1\ell m} \mathrm{e}^{\mathrm{i}n(k_0 r - h\cos\theta)} \times [1 + (-1)^{\ell+m} e^{2\mathrm{i}nh\cos\theta}] \boldsymbol{V}_{\ell m}(\cos\theta, \phi). \quad (105)$$

Expressions for the magnetic field can be found in the same way, and it turns out that in the far field we have

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{n}{c} \hat{\boldsymbol{r}} \times \boldsymbol{E}(\boldsymbol{r}). \quad (106)$$

To verify this, one needs to use Eq. (75) or (76) in the form

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{V}_{\ell m}(\cos\theta, \phi) = 0, \quad (107)$$

from which we also find $\hat{\boldsymbol{r}} \cdot \boldsymbol{E}(\boldsymbol{r}) = 0$ in the radiation zone.

The Poynting vector for time-harmonic fields is defined as

$$\boldsymbol{S}(\boldsymbol{r}) = \frac{1}{2\mu_0} \mathrm{Re} \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})^* \quad (108)$$

and with Eq. (106) this simplifies to

$$\boldsymbol{S}(\boldsymbol{r}) = \frac{n}{2\mu_0 c} [\boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r})^*] \hat{\boldsymbol{r}}. \quad (109)$$

The emitted power P per unit solid angle Ω is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = r^2 \boldsymbol{S}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}}, \quad (110)$$

which can now easily be evaluated. We shall normalize the result as

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = P_1 \mathcal{N}_{\eta\ell m}(\hat{\boldsymbol{r}}), \quad (111)$$

where P_1 is the total power emitted by a multipole embedded in a medium with index of refraction n, but without any boundaries present. This power is given by

$$P_1 = \frac{1}{2n\mu_0 c} \left(\frac{k_0^2}{4\pi\varepsilon_0} \right)^2 |b_{\eta\ell m}|^2. \quad (112)$$

The normalized intensity distribution then becomes

$$\mathcal{N}_{\eta\ell m}(\hat{\boldsymbol{r}}) = 2[1 + \eta(-1)^{\ell+m} \cos(2nh\cos\theta)] \times \boldsymbol{V}_{\ell m}(\cos\theta, \phi) \cdot \boldsymbol{V}_{\ell m}(\cos\theta, \phi)^*. \quad (113)$$

It follows from Eq. (63) that $\mathcal{N}_{\eta\ell m}(\hat{\boldsymbol{r}})$ is independent of angle ϕ, and from Eqs. (65)–(67) that $\mathcal{N}_{\eta\ell m}(\hat{\boldsymbol{r}})$ is independent of the sign of m.

In order to see the significance of the very simple result (113), we first recall that without the perfect conductor present, the normalized intensity distribution is given by

$$\mathcal{N}_{\eta\ell m}(\hat{\boldsymbol{r}}) = \boldsymbol{V}_{\ell m}(\cos\theta, \phi) \cdot \boldsymbol{V}_{\ell m}(\cos\theta, \phi)^*, \quad (114)$$

which holds for a 4π solid angle. From the properties of the vector spherical harmonics it can be shown that the integrated $\mathcal{N}_{\eta\ell m}(\hat{\boldsymbol{r}})$ from Eq. (114) equals unity. The effect of the interface is first that an overall factor of 2 appears. This is due to the fact that the radiation emitted by the multipole in the negative z-direction reflects at the interface, and contributes to the power in $z > 0$. The term with $\cos(2nh\cos\theta)$ in square brackets accounts for the modulation due to interference between the directly-emitted multipole waves in the positive z-direction and the reflected waves. When the distance between the multipole and the interface is much larger than a wavelength, this term effectively averages out to zero (when integrated over a small $\Delta\Omega$), and the resulting distribution is just twice that of Eq. (114). Fig. 3 shows the difference between the free-space radiation pattern and the interference pattern for the case of a magnetic dipole with $m = 0$.

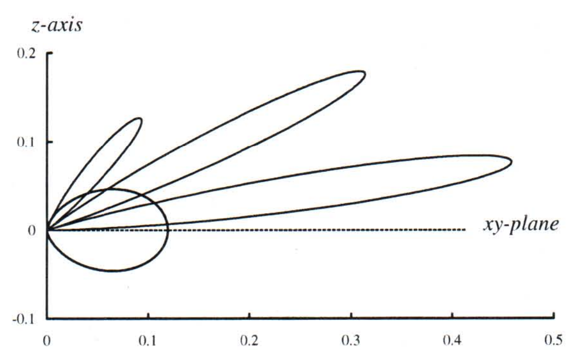

Fig. 3. Polar diagram of the radiation pattern of a magnetic dipole with $m = 0$, for $n = 1$. The dimensionless distance between the dipole and the interface is $h = 3\pi$, corresponding to a separation of 1.5 wavelengths. The thick curve is the emission pattern in free space and the thin line is the intensity distribution near the perfect conductor.

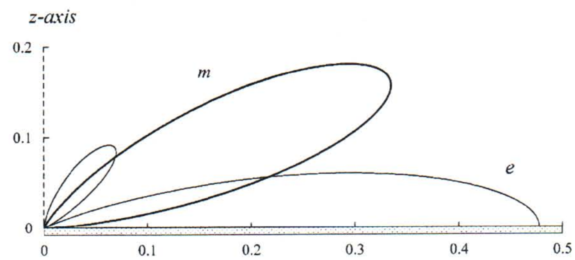

Fig. 4. Radiation pattern of an electric (e) and a magnetic (m) dipole near a perfect conductor, both with $m = 0$, and for $n = 1$. Here we took $h = \pi$, corresponding to a separation of half a wavelength. The figure illustrates that at the angle where constructive interference for the magnetic dipole occurs, the radiation pattern for the electric dipole has a minimum, and vice versa.

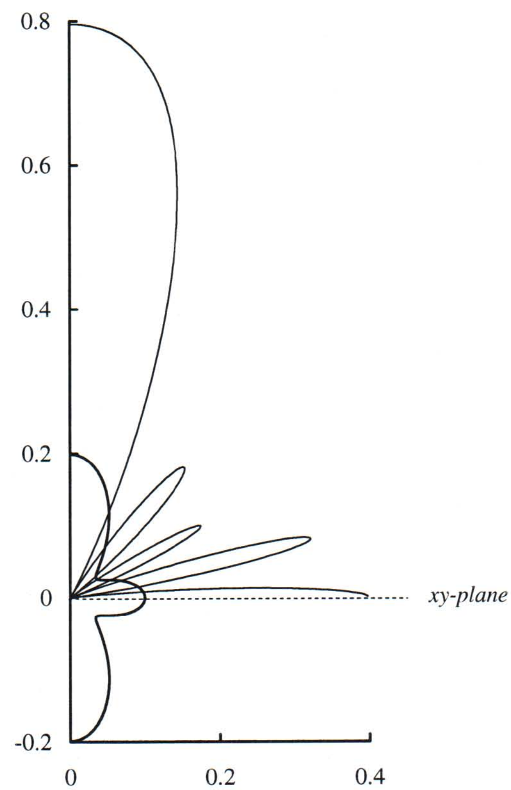

Fig. 5. The figure shows the angular intensity distribution of the radiation emitted by an electric quadrupole in free space (thick line) and near the perfect conductor (thin line) for $m = 1$, $n = 1$ and $h = 4\pi$. For emission in the positive z-direction, the intensity of the radiation emitted by the quadrupole near the perfect conductor is four times the value of the intensity of emission by the same quadrupole in free space.

The factor $\eta(-1)^{\ell+m}$ preceding the interference factor $\cos(2nh\cos\theta)$ is either 1 or -1, indicating that when for a certain m value the interference is constructive for a given angle θ, than for an m value one higher or lower the interference is destructive. It also shows that when the interference is constructive for an electric multipole for a given θ, then for a magnetic multipole with the same m value the interference is destructive at this angle, and vice versa. This is illustrated in Fig. 4 for an electric and magnetic dipole. Finally, when $\eta(-1)^{\ell+m} = +1$, then the intensity at $\theta = 0$ is four times the value of the intensity without the interface. This dramatic increase in intensity is shown in Fig. 5 for an electric quadrupole with $m = 1$.

13. Conclusions

We have obtained angular spectrum representations of the transverse vector multipole fields, given by Eqs. (36) and (38), and thereby of the electromagnetic multipole fields. The angular spectra could be expressed in terms of generalized vector spherical harmonics $\boldsymbol{V}_{\ell m}(\pm\cos\bar{\theta}, \bar{\phi})$, which in turn were expressed in generalized spherical harmonics. The result is a superposition of plane waves with wave vectors $\boldsymbol{K}_{\pm}$, which have their parallel parts $\boldsymbol{k}_{\parallel}$ as free parameters (integration variable). Depending on the value of $\boldsymbol{k}_{\parallel}$, the wave is either traveling or evanescent. For a traveling wave, the angles $(\bar{\theta}, \bar{\phi})$ appearing in the arguments of the generalized vector spherical harmonics are the spherical-coordinate angles of wave vector $\boldsymbol{K}_{+}$. The wave with wave vector $\boldsymbol{K}_{-}$ travels in the opposite direction with respect to the xy-plane, as shown in Fig. 1, and its direction is given by the angles $(\pi - \bar{\theta}, \bar{\phi})$. For an evanescent wave, the variable $\cos\bar{\theta}$ becomes imaginary.

The angular spectrum representation of the multipole fields is particularly useful for the study of reflection and refraction by a plane interface. As an example, we have considered the reflection at a perfect conductor. It appeared that the reflected field of a multipole is the multipole field itself, apart from a possible phase factor, but its source location is shifted to the mirror position below the surface. An electric or magnetic multipole of order ℓ has $2\ell+1$ independent m components ('magnetic quantum number'). The reflected field can be considered as being generated by an image source, and the phase factor can be assigned to the current distribution of this image source. The phase factor (+1 or −1) depends on the value of m and on the type of multipole field (electric or magnetic). When a source consists of a current distribution which is a superposition of various m values, then the various m-components acquire a different phase factor. From this it follows that for the mirror image of an electric dipole the parallel component changes sign, as compared to the source dipole, whereas for a magnetic dipole the perpendicular component changes sign, which is a well-known fact. The method can be extended to multipoles of arbitrary order. For instance, the mirror image of an electric quadrupole is given by Eq. (101).

With the method of stationary phase, an asymptotic approximation of the angular spectrum representations can be obtained, and we have applied this to evaluate the intensity distribution of a multipole near a perfect conductor. The normalized intensity distribution is given by Eq. (113). The result is the same distribution as in free space, Eq. (114), modified by an interference factor (factor in square brackets) and a factor of two. The difference in radiation pattern between an electric and a magnetic multipole of the same order (ℓ, m) is a simple minus sign in the expression (the parameter η) A consequence of this minus sign is that when an electric multipole exhibits constructive interference at a certain observation angle then the magnetic multipole of the same order shows destructive interference at this angle, and vice versa.

Appendix A

A vector $\boldsymbol{V}$ can be expanded onto the set of spherical unit vectors as

$$\boldsymbol{V} = \sum_q V_q \boldsymbol{e}_q^* \tag{A.1}$$

and since $\boldsymbol{e}_q^* \cdot \boldsymbol{e}_{q'} = \delta_{qq'}$, the spherical components V_q of $\boldsymbol{V}$ are

$$V_q = \boldsymbol{e}_q \cdot \boldsymbol{V}. \tag{A.2}$$

With $\boldsymbol{e}_q^* = (-1)^q \boldsymbol{e}_{-q}$, Eq. (A.1) can also be written as

$$\boldsymbol{V} = \sum_q (-1)^q V_{-q} \boldsymbol{e}_q. \tag{A.3}$$

Comparison with Eq. (49) then shows that for electric dipole radiation with a given m value, the dipole moment $\boldsymbol{d}$ has only one non-vanishing spherical component (with $q = -m$), and the same holds for the magnetic dipole.

For electric quadrupole radiation the current source involves the quadrupole tensor $\overleftrightarrow{Q}$, for which a similar decomposition can be made. To this end, we note that the significance of the spherical components of a vector is that these are the components of a spherical (irreducible) tensor of rank 1. According to a well-known theorem [29], two vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ can be "multiplied" to form an irreducible tensor of rank k, with $k = 0$, 1 or 2. The qth component of the irreducible tensor of rank k is defined as

$$T_q^{(k)} = \sum_{\mu\mu'} (1\mu 1\mu'|kq) V_\mu W_{\mu'} \tag{A.4}$$

for $q = -k, \ldots, k$. We now make a slight generalization of this theorem as follows. We can write $V_\mu W_{\mu'}$ as $\boldsymbol{e}_\mu \cdot (\boldsymbol{VW}) \cdot \boldsymbol{e}_{\mu'}$, and then we note that the dyad $\boldsymbol{VW}$ is a Cartesian second rank tensor. If we replace $\boldsymbol{VW}$ in Eq. (A.4) by an arbitrary Cartesian second rank tensor $\overleftrightarrow{T}$, then the combination of matrix elements

$$T_q^{(k)} = \sum_{\mu\mu'} (1\mu 1\mu'|kq) \boldsymbol{e}_\mu \cdot \overleftrightarrow{T} \cdot \boldsymbol{e}_{\mu'} \tag{A.5}$$

transforms under rotations in the same way as the right-hand side of Eq. (A.4), and therefore the quantities $T_q^{(k)}$ defined in this way are also the qth components of an irreducible tensor of rank k. Eq. (A.5) can be inverted with the orthonormality of the Clebsch–Gordan coefficients according to

$$\boldsymbol{e}_\mu \cdot \overleftrightarrow{T} \cdot \boldsymbol{e}_{\mu'} = \sum_{kq} (1\mu 1\mu'|kq) T_q^{(k)}. \tag{A.6}$$

Any tensor $\overleftrightarrow{T}$ can be written as

$$\overleftrightarrow{T} = \sum_{\mu\mu'} (\mathbf{e}_\mu \cdot \overleftrightarrow{T} \cdot \mathbf{e}_{\mu'}) \mathbf{e}^*_\mu \mathbf{e}^*_{\mu'} \quad (A.7)$$

and combination with Eq. (A.6) yields

$$\overleftrightarrow{T} = \sum_{kq\mu\mu'} (1\mu 1\mu'|kq) T^{(k)}_q \mathbf{e}^*_\mu \mathbf{e}^*_{\mu'}. \quad (A.8)$$

Then we define the "unit tensors"

$$\overleftrightarrow{E}^{(k)}_q = \sum_{\mu\mu'} (1\mu 1\mu'|kq) \mathbf{e}_\mu \mathbf{e}_{\mu'} \quad (A.9)$$

in terms of which Eq. (A.8) becomes

$$\overleftrightarrow{T} = \sum_{kq} T^{(k)}_q \overleftrightarrow{E}^{(k)^*}_q , \quad (A.10)$$

since the Clebsch–Gordan coefficients are real. This expansion is equivalent to the expansion of vector $\mathbf{V}$ in spherical unit vectors, as given by Eq. (A.1). The Clebsch–Gordan coefficients have the property $(1-\mu 1-\mu'|kq) = (-1)^k (1\mu 1\mu'|k-q)$, from which we have

$$\overleftrightarrow{E}^{(k)^*}_q = (-1)^{k+q} \overleftrightarrow{E}^{(k)}_{-q} \quad (A.11)$$

in analogy to $\mathbf{e}^*_q = (-1)^q \mathbf{e}_{-q}$, and this allows us to write Eq. (A.10) in the alternative form

$$\overleftrightarrow{T} = \sum_{kq} (-1)^{k+q} T^{(k)}_{-q} \overleftrightarrow{E}^{(k)}_q , \quad (A.12)$$

as in Eq. (A.3) for vectors. Comparison with Eq. (56) shows that electric quadrupole radiation for a given m has a source with a quadrupole tensor identical in form to Eq. (A.12), but with only the term $k = 2$ and $q = m$ present.

Eq. (A.5) expresses $T^{(k)}_q$ in terms of the matrix elements $\mathbf{e}_\mu \cdot \overleftrightarrow{T} \cdot \mathbf{e}_{\mu'}$ of $\overleftrightarrow{T}$, and these can in turn be expressed in terms of the Cartesian components $T_{\alpha\beta}$ with $\alpha, \beta = x, y, z$. The results are listed on p. 174 of Ref. [30]. In particular, if $\overleftrightarrow{T}$ is a symmetric Cartesian tensor it follows by inspection that the three spherical components with $k = 1$ vanish. The approach with the unit tensors does not appear to be readily available in the literature, so here we list these tensors explicitly for reference:

$$\overleftrightarrow{E}^{(0)}_0 = \frac{1}{\sqrt{3}} (\mathbf{e}_1 \mathbf{e}_{-1} + \mathbf{e}_{-1} \mathbf{e}_1 - \mathbf{e}_0 \mathbf{e}_0) = -\frac{1}{\sqrt{3}} \overleftrightarrow{I}, \quad (A.13)$$

$$\overleftrightarrow{E}^{(1)}_{\pm 1} = \pm \frac{1}{\sqrt{2}} (\mathbf{e}_{\pm 1} \mathbf{e}_0 - \mathbf{e}_0 \mathbf{e}_{\pm 1}), \quad (A.14)$$

$$\overleftrightarrow{E}^{(1)}_0 = \frac{1}{\sqrt{2}} (\mathbf{e}_1 \mathbf{e}_{-1} - \mathbf{e}_{-1} \mathbf{e}_1), \quad (A.15)$$

$$\overleftrightarrow{E}^{(2)}_{\pm 2} = \mathbf{e}_{\pm 1} \mathbf{e}_{\pm 1}, \quad (A.16)$$

$$\overleftrightarrow{E}^{(2)}_{\pm 1} = \frac{1}{\sqrt{2}} (\mathbf{e}_{\pm 1} \mathbf{e}_0 + \mathbf{e}_0 \mathbf{e}_{\pm 1}). \quad (A.17)$$

$$\overleftrightarrow{E}^{(2)}_0 = \frac{1}{\sqrt{6}} (\mathbf{e}_1 \mathbf{e}_{-1} + \mathbf{e}_{-1} \mathbf{e}_1 + 2\mathbf{e}_0 \mathbf{e}_0). \quad (A.18)$$

References

[1] M.E. Rose, Multipole Fields, Wiley, New York, 1955.
[2] I.I. Sobelman, Atomic Spectra and Radiative Transitions, Springer, Berlin, 1992, p. 216.
[3] J.M. Blatt, V.F. Weisskopf, Theoretical Nuclear Physics, Dover, New York, 1991, Chapter 12.
[4] W. Lukosz, R.E. Kunz, J. Opt. Soc. Am. 67 (1977) 1607.
[5] W. Lukosz, R.E. Kunz, J. Opt. Soc. Am. 67 (1977) 1615.
[6] H.F. Arnoldus, J.T. Foley, J. Opt. Soc. Am. A 21 (2004) 1109.
[7] K.H. Drexhage, Prog. Opt. XII (1974) 163.
[8] G.S. Agarwal, Phys. Rev. Lett. 32 (1974) 703.
[9] R.R. Chance, A. Prock, R. Silbey, Adv. Chem. Phys. 39 (1978) 1.
[10] G.W. Ford, W.H. Weber, Surf. Sci. 109 (1981) 451.
[11] P. Goy, J.M. Raimond, M. Gross, S. Haroche, Phys. Rev. Lett. 50 (1983) 1903.
[12] G.W. Ford, W.H. Weber, Phys. Rep. 113 (1984) 195.
[13] R.G. Hulet, E.S. Hilfer, D. Kleppner, Phys. Rev. Lett. 55 (1985) 2137.
[14] W. Jhe, A. Anderson, E.A. Hinds, D. Meschede, L. Moi, S. Haroche, Phys. Rev. Lett. 58 (1987) 666.
[15] D.J. Heinzen, J.J. Childs, J.E. Thomas, M.S. Feld, Phys. Rev. Lett. 58 (1987) 1320.
[16] J. Gasper, G.C. Sherman, J.J. Stamnes, J. Opt. Soc. Am. 66 (1976) 955.
[17] J.E. Sipe, Surf. Sci. 105 (1981) 489.
[18] J.E. Sipe, J. Opt. Soc. Am. B 4 (1987) 481.
[19] A. Erdélyi, Physica 4 (1937) 107.
[20] A.J. Devaney, E. Wolf, J. Math. Phys. 15 (1974) 234.
[21] L. Mandel, E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge, England, 1995, Sec. 3.2.4.
[22] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcedental Functions, vol. I, Krieger, Florida, 1981, Chapter 3.

[23] J.D. Jackson, Classical Electrodynamics, third ed, Wiley, New York, 1999, p. 429.
[24] J.M. Eisenberg, W. Greiner, Excitation Mechanisms of the Nucleus, North-Holland, Amsterdam, The Netherlands, 1970, Chapter 3.
[25] M.E. Rose, Elementary Theory of Angular Momentum, Dover, New York, 1995, Chapter 7.
[26] J. van Bladel, Singular Electromagnetic Fields and Sources, Clarendon, Oxford, 1991, Chapter 2.
[27] M. Born, E. Wolf, Principles of Optics, seventh (expanded) ed, Cambridge University Press, Cambridge, UK, 1999. App. III, p. 890.
[28] G.C. Sherman, J.J. Stamnes, É. Lalor, J. Math. Phys. 17 (1976) 760.
[29] J.J. Sakurai, Modern Quantum Mechanics, Revised Edition, Addison-Wesley, Reading, Massachusetts, 1994, p. 237.
[30] M. Weissbluth, Atoms and Molecules, Academic Press, New York, 1978, p. 174.